Electromagnetics Made Easy

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Electromagnetics Made Easy

B.Srinivasan

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Preface

Electromagnetics / Electromagnetic theory is one of the fundamental theory which explains number of naturally occurring phenomena and lead to technological innovations. Hence it is not surprising to see that it is one of the fundamental subject for Electronics / Electrical / Telecommunication engineering students and physics postgraduates. However with the concept of vectors which involve directions, and intangible electric and magnetic fields students find it hard to understand the subject.

Irrespective of whichever institution and country the student studies, one of the information to quickly reach him, after joining the course will be, Electromagnetics / Electromagnetic theory is a difficult subject and understanding the concepts will be tedious. Getting a pass in the paper is even more harder. Because of this reason students aim a minimum pass in the subject and not to gain marks.

An attempt has been made in this book to change the above scenario. This book is a result of a decade of handling the subject to engineering students and physics postgraduates. The subject will be explained in a very very easy manner so that the difficulty in learning the subject is completely eliminated. Apart from helping students to pass their university exam this book will guide aspiring students for clearing their competitive exams. The student will find the subject very very interesting if he goes through the book seriously.

As the reader will observe, the concepts are explained sequentially. As an example, in electrostatics Coulomb's law is followed by Gauss law and then by potential formulation giving reasons for developing each method.

The backbone of electromagnetics is vector analysis. Hence chapter 1 is completely devoted to vector analysis. The first fifteen pages will give the reader a flavor of, what a vector is and why we reduce a vector problem into a scalar problem. In the same chapter fundamental concepts like flux, divergence, curl, gradient, different coordinate systems etc are explained in detail. Chapter 2 to Chapter 7 can be grouped under three major sections – Electrostatics , magnetostatics and electromagnetic fields.

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Electrostatics primarily concerns with the calculation of electric field for a given static charge distribution. Different methods for calculation of electric field for static charge distributions is discussed and the reason for developing each method is explained. Dielectrics and capacitors are then described in detail.

Magnetostatics concentrates on calculation of magnetic field for steady currents. Different methods for calculation of magnetic field is described and compared with their electrostatic counterpart. Finally their application in inductors and magnetic circuits in elaborated.

The third section electromagnetic field discusses about fields under dynamic conditions. Starting with Faradays laws of induction then followed by Maxwells equation, each concept is explained clearly. Then introducing electromagnetic waves and their propagation in different media, reflection and transmission of electromagnetic wave is elaborated.

Each chapter contains solved problems and a list of exercises. Your feedback is highly valuable and suggestions for the improvement of the book is appreciated. Feedback, suggestions and any corrections (printing errors or any other mistake) can be mailed to correction.emt@gmail.com.

– The Author.

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Chapter 1

Vector Analysis

1.1 Introduction

a)

Several complex problems in electromagnetism involve many physical quantities having both magnitude and direction. These quantities called vectors play an important role in understanding the multitude of phenomena in electrostatics, magnetostatics etc.

A scalar is a quantity which involves magnitude alone. A vector is a quantity which involves both magnitude and direction. Examples of scalar quantities are mass, length, temperature etc., Examples of vector quantities are displacement, velocity, force etc.

The following example will make the difference between vector and scalar much clearer. Suppose, assume that there are two persons; Person A and person B trying to add rice in a empty box as shown in figure 1.1a and figure 1.1b. Person A is having 7 kg of rice and person B is having 2 kg of rice.

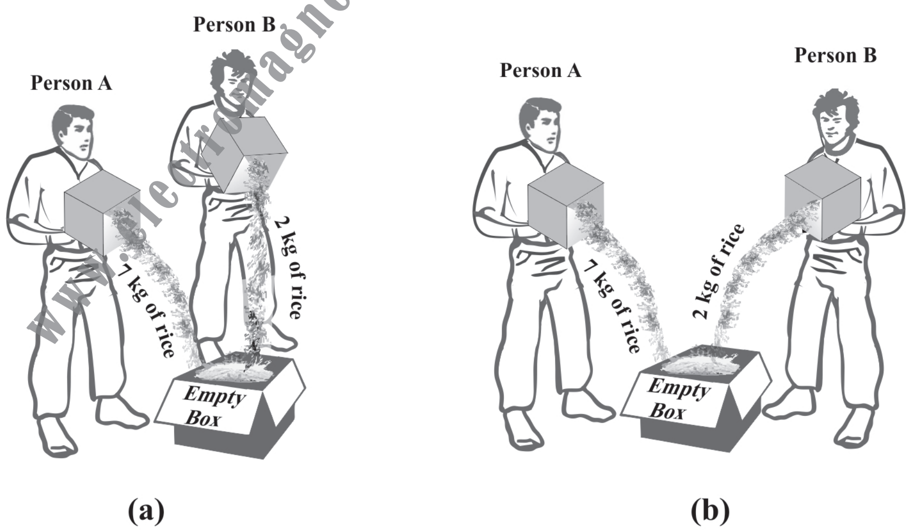
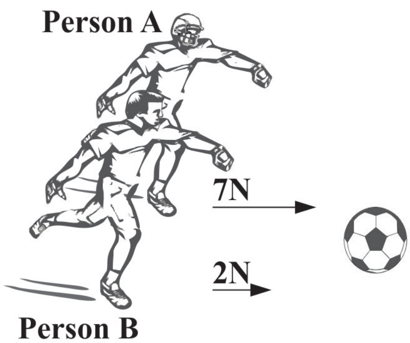
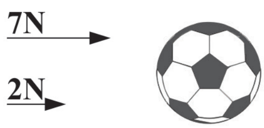


Fig. 1.1

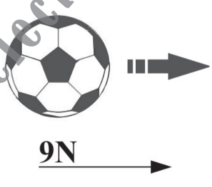
Vector Analysis



Person A and person B standing adjacent to each other and kicking the football. with a force of 7N, 2N respectively.

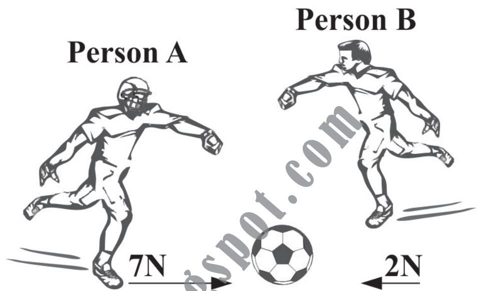


Forces 7N, 2N acting in the same direction on the football.



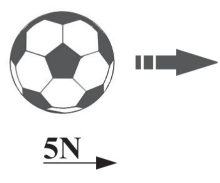
The net force 9N acting on the football with the resultant direction shown in the figure.

(a)



Person A and person B standing opposite to each other and kicking the football with the force of 7N, 2N respectively.

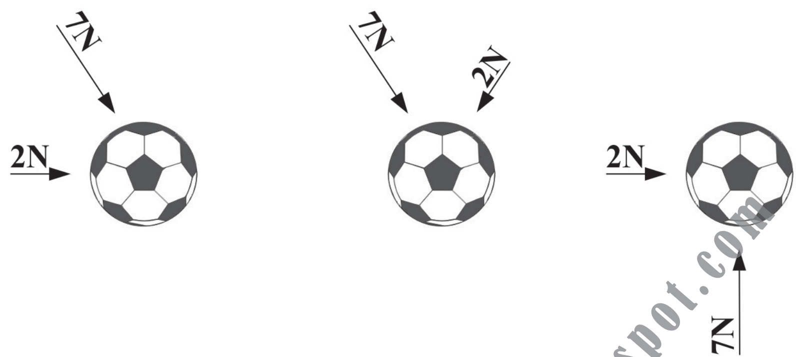
Forces 7N, 2N acting in opposite direction on the football.



The net force 5N acting on the football with the resultant direction as shown in the figure.

(b)

Fig.1.2



Person A and person B standing in different possible directions and kicking the football.

Fig.1.2c

In figure 1.1a person A and person B are standing in same direction and are adding the rice in the empty box. After adding the rice in the box, the box would contain 9 kg of rice. In figure 1.1 b person A and person B are standing opposite to each other and are adding the rice in the empty box. Once again the box would then contain 9 kg of rice after the rice has been added in the box. So, the conclusion is, whatever direction the mass is added the net effect is same, we get 9 kg of rice in total. The total (9 kg in our example) simply depends on the initial ***magnitude*** (7, 2 kg's in our example) of the masses and not on the direction in which the masses are added.

Now consider figure 1.2a and 1.2b. In figure 1.2a person A and person B are kicking a football with a force of 7N and 2N (Numbers are not to be taken seriously) in the same direction. That is person A and person B are standing adjacent to each other and kicking the football. The total effect or the total force is 7N + 2N = 9N and the ball will move in the direction of the kick. In figure, 1.2b person A and person B are standing opposite to each other and kicking the football with a force of 7N and 2N respectively. The net or total force is 7N - 2N = 5N acting in the direction of kick of person A. Now, the conclusion is the total force (9N, 5N in our example) depends on, not only on the ***magnitude*** (7N, 2N in our example) but also on the ***direction*** of the initial forces. Person A and person B can stand in any possible direction and kick the football. Few such possible directions are shown in figure 1.2c. For such situations shown in figure 1.2c the net or total force and its direction must be obtained by using laws of vector addition.

Thus, irrespective of whatever direction the initial masses are added we get the same total mass. However when forces are added depending upon the direction of the initial forces, we get a different net or total force.

Quantities which behave like mass are called scalars and quantities which behave like force are called vectors. Thus, the above example clearly distinguishes between a vector and scalar.

We have another important information from the above example. During addition of the scalars one need not bother about their direction. One must take into account their magnitude alone. Because irrespective of whatever direction the scalars (masses in our example) are added we get the same total or net scalar (mass in our example) we need not bother about their direction.

However, during addition of vectors you need to keep track of not only their magnitude, but also their direction. In our force example in figure 1.2a, 1.2b, 1.2c the initial direction of 7N, 2N must always be tracked if we want to get the total or net force.

As we will see shortly the direction of the total or net force must also be tracked. Because the direction of the initial vector [7N, 2N in our force example in figure 1.2a, 1.2b, 1.2c] must always be tracked in order to obtain the value of the direction and magnitudes of the total or net vector ***vector addition becomes complicated than scalar addition***. Scalar addition is simple. One need to bother about their magnitude alone not their direction. In our mass example in figure 1.1, 7 kg, 2 kg, can be added either in same direction or opposite direction or any possible direction we still get the same total or net mass 9 kg. So, no need to worry about direction of initial masses. The same is true for any scalar.

However vector addition is comparatively complicated than scalar addition. In our force example in figure 1.2a, 1.2b, 1.2c the total or net force depends upon the direction and magnitude of the initial forces 7N, 2N. Hence, in addition to magnitude, the direction of initial forces needs to be always tracked. The same is true for any vector. This makes vector addition complicated.

In order to get a feel how much difficult it is to deal with vectors, assume that you are interested in adding ten masses m_1, m_2, ... m_{10} and ten forces $\mathbf{F_1}$, $\mathbf{F_2}$ .... $\mathbf{F_{10}}$. For all ten masses you need not keep track of their direction. Just add $m_1 + m_2 + ... + m_9 + m_{10}$. But what about forces. Each of the ten force's is going

to have its own direction and you have to mess up with the direction. Now you can realize how much difficult it is to deal with vectors.

Similar to vector addition, vector multiplication, vector differentiation, vector integration etc., become difficult because vectors involve direction. For example if you want to multiply two scalars then you simply take their magnitude and multiply them. However, in the case of multiplication of two vectors, we have two types - dot product and cross product, as vectors involve direction. Vector multiplication thus is comparatively complicated than scalar multiplication.

In example 1.10 the difficulty involved in working with vectors is further explained.

Hence, in electromagnetics wherever possible our aim will be to reduce a vector problem into a scalar problem because all operations can be carried out easily with scalars than with vectors.

The ball example which we have shown in figure 1.2 must not be taken too seriously. Clearly the ball will hit person B and stop. In the case of electromagnetics we will be discussing with the invisible electric and magnetic forces where this problem doesn't arise. As an example, in figure 1.2 replace the ball with a particle of charge q and the persons, with electric forces acting in the respective directions. We have taken the ball example – the real life example for better grasping of the concept of vectors for the reader.

In figures 1.1, 1.2 we have used scalar addition (normal usual addition) and vector addition, to differentiate between a scalar and vector which the reader will not find it in any book. In standard books the vector addition will be defined in the way, we have defined in section 1.4. The reader can use parallelogram law of vectors (which is not described in this book) to check whether we get similar results for figure 1.2.

A final point to be noted is some students raise an objection that if vectors are difficult to deal with, then why we are considering force as a vector. Why force cannot be considered as a scalar. Kindly note that the direction dependency of the force shown in figure 1.2a, 1.2b is not our wish. We are not forcing the force to act in such a manner; adding and cancelling depending upon direction. Direction dependency of force (vectors) is naturally and intrinsically present in it and direction independency of mass (scalars) is naturally and intrinsically present in it.

b)

Previously we noted that not only the direction of the initial forces must be tracked but the direction of the net or total force must also be tracked. In order to understand the above point consider figure 1.2d. In figure 1.2d person A, B are kicking the food ball exactly opposite to the manner shown in figure 1.2b. Clearly, in figure 1.2d the resultant force is also 5N but now it is opposite to the resultant force 5N in figure 1.2b. Thus, the resultant force in figure 1.2b, 1.2d are 5N but acting completely in opposite direction. The result is, football in figure 1.2d moves completely in opposite direction as compared to figure 1.2b. Thus although the total force in figure 1.2b, 1.2d are 5N the result they produce is different [football moving in opposite direction] because the direction in which the resultant force 5N acting in figure 1.2b, 1.2d are exactly opposite.

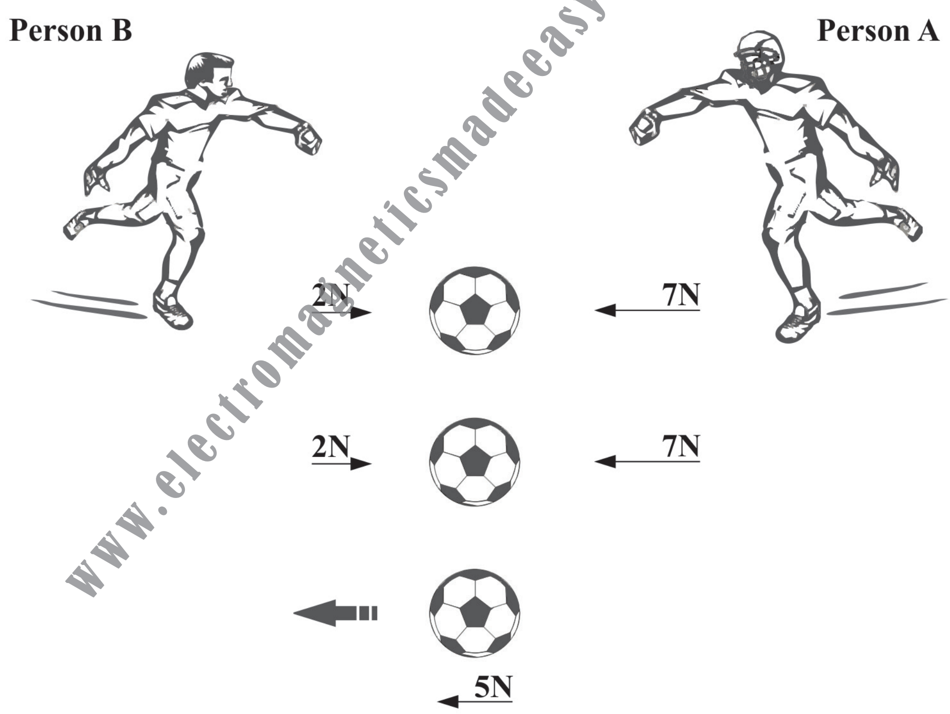


Fig.1.2d

Thus the conclusion is, although the magnitude of two forces are same if their directions are different they are completely different forces. The same is true for any vector.

However, we need not track the direction of total or net mass 9 kgs in figure 1.1a, 1.1b because mass doesn't involve direction. The same is true for any scalar.

In you lower classes you would have learnt how to add, subtract, multiply differentiate and integrate scalars. It's very easy. Now in the coming chapters we will learn how add, subtract, multiply, differentiate and integrate vectors.

1.2 Graphical Representation of Vectors

A vector is graphically represented by an arrow. The length of the arrow represents the magnitude of the vector with respect to a pre-chosen scale. The direction of the vector is given by the arrow head. For example figure 1.3 shows the

Fig.1.3

graphical representation of a vector **A**. The arrow pointing from O to P represents the direction of the vector and the length represents the magnitude (to a suitable scale) of the vector. The end P containing the arrow head is called the head of the vector whereas the other end O is called the tail of the vector. As for an example in figure 1.3 if **A** is the velocity say for example 20km/hr and in figure 1.3 if 1cm corresponds to 5km/hr then the length of OP will be 4cms which specifies a velocity of 20km/hr.

1.3 Symbolic Representation of Vectors

The usual representation of a vector is in bold letters like **A** or a letter with an arrow on it $\vec{A}$. The magnitude of a vector is usually represented by |**A**| or A without bold letter. In this book we will follow the notation bold letters **A** for the vectors and A without bold letter for its magnitude. Care must be taken by the reader to properly identify the vector notation. For example **A** in this book refers to a vector in both magnitude and direction. On the other hand writing the above **A** without bold face letters like A means that we are mentioning only the magnitude of the vector.

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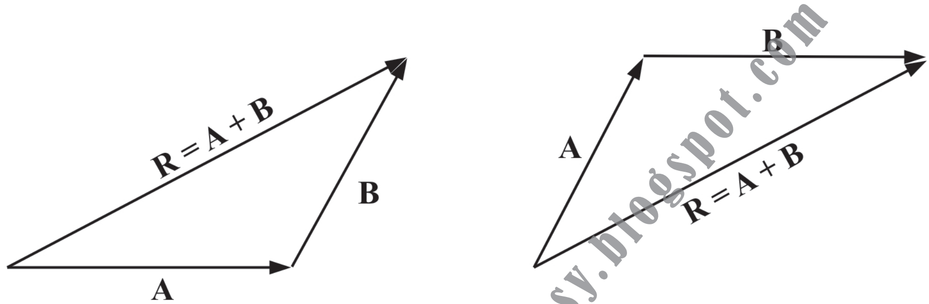
Vector Analysis

1.4 Vector Addition

Figure 1.2 illustrates addition of vectors is not as simple as addition of scalars. In figure 1.4a the addition of two vectors **A** and **B** is shown. The tail of the **B** is placed on the head of **A**. The resultant **R** is given by the vector drawn from the tail of the vector **A** to the head of the vector **B**.



(a) (b)

Fig.1.4

In figure 1.4a we add vector **B** to **A**. In figure 1.4b we add vector **A** to **B**. In both the cases the resultant is same. Hence vector addition is commutative.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.1}$$

Also vector addition is associative

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{1.2}$$

1.5 Subtraction of Vectors

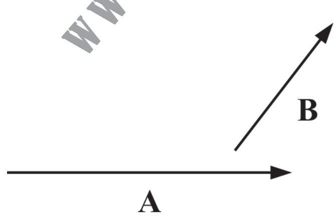


Fig.1.5

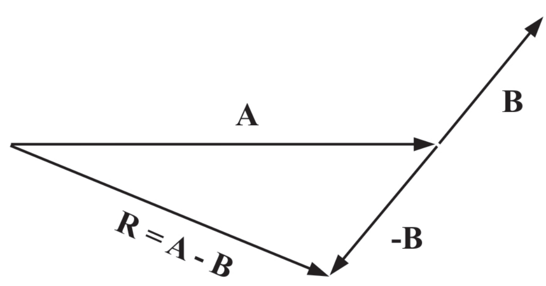


Fig.1.6

Figure 1.5 shows two vectors **A**, **B**. For subtracting **B** from **A** reverse the direction of **B** (without changing the magnitude) as shown in figure 1.6 and then add **B** to **A**.

1.6 Multiplication of a Vector by a Scalar

If a vector is multiplied by a scalar u then the magnitude of the vector is increased by u times and the direction of the vector remains unchanged. If u=3 then

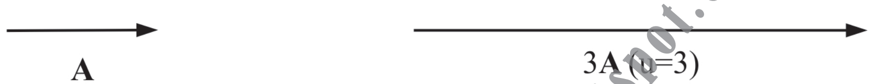


Fig.1.7

the magnitude of vector is increased by 3 times as shown in figure 1.7. On the other hand if u is negative then the direction of the vector is reversed.

Multiplication of vector by a scalar is distributive.

$$u\,(\mathbf{A} + \mathbf{B}) = u\mathbf{A} + u\mathbf{B} \tag{1.3}$$

1.7 Multiplication of Vectors: Dot Product of Two Vectors

The dot product of two vectors **A**, **B** is defined by

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \tag{1.4}$$

θ is the smallest angle between the two vectors when they are placed tail to tail. Dot product is also sometimes called scalar product because the result of a dot product is a scalar. Dot product obeys commutative and distributive laws.

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1.5}$$

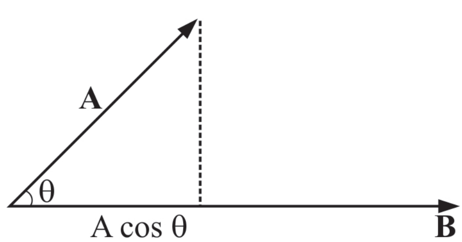


Fig.1.8

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{1.6}$$

The geometrical meaning of the dot product can be understood by the aid of figure 1.8

As can be seen from figure 1.8 the projection of **A** on **B** is $A\cos\theta$. Geometrically the dot product **A**. **B** is the product of B and the projection of **A** on **B**.

If two vectors are parallel then **A** . **B** = $AB\cos 0$ = AB and if they are perpendicular then **A** . **B** = AB cos 90 = 0.

1.8 Multiplication of Vectors - Cross - Product of Two Vectors

The cross product of two vectors is denoted by **A** × **B** and is defined by

$$\mathbf{A} \times \mathbf{B} = AB\sin\theta \; \hat{\mathbf{n}} \tag{1.7}$$

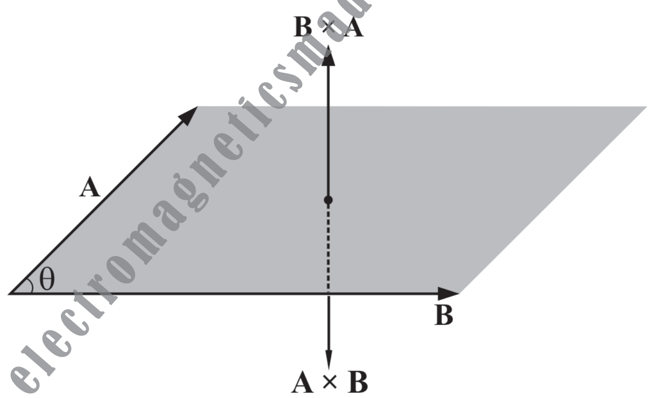
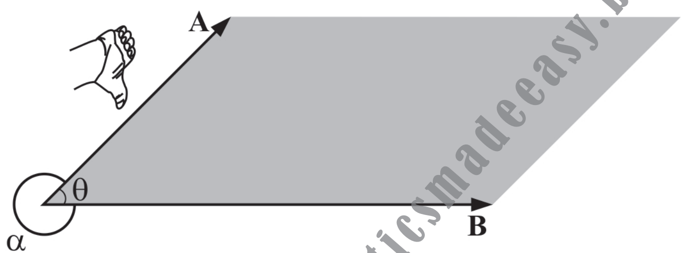


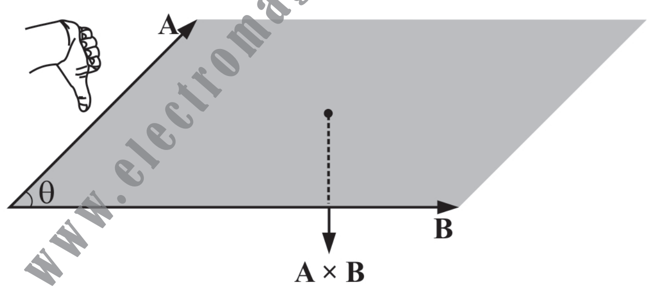
Fig.1.9

Here $\hat{\mathbf{n}}$ is the unit vector in the direction perpendicular to the plane containing **A**, **B**. However there are two perpendicular directions to the plane as shown in figure 1.9. The proper direction of $\hat{\mathbf{n}}$ is given by right hand rule. Let all the fingers of the right hand except the thumb point in the direction of **A** and let the fingers curl towards **B** through the small angle between **A** and **B**. Then the thumb gives the direction of $\hat{\mathbf{n}}$ or the direction of **A** × **B**. The entire sequence is shown in figure 1.10a. Similar sequence for obtaining the direction of **B** × **A** is shown in figure

Right hand pointing in the direction of **A.** Smaller angle is θ. Larger angle is α.



Fingers curling from **A** to **B** through the smaller angle θ.



Thumb gives the direction of **A** × **B.**

Fig.1.10a

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Vector Analysis

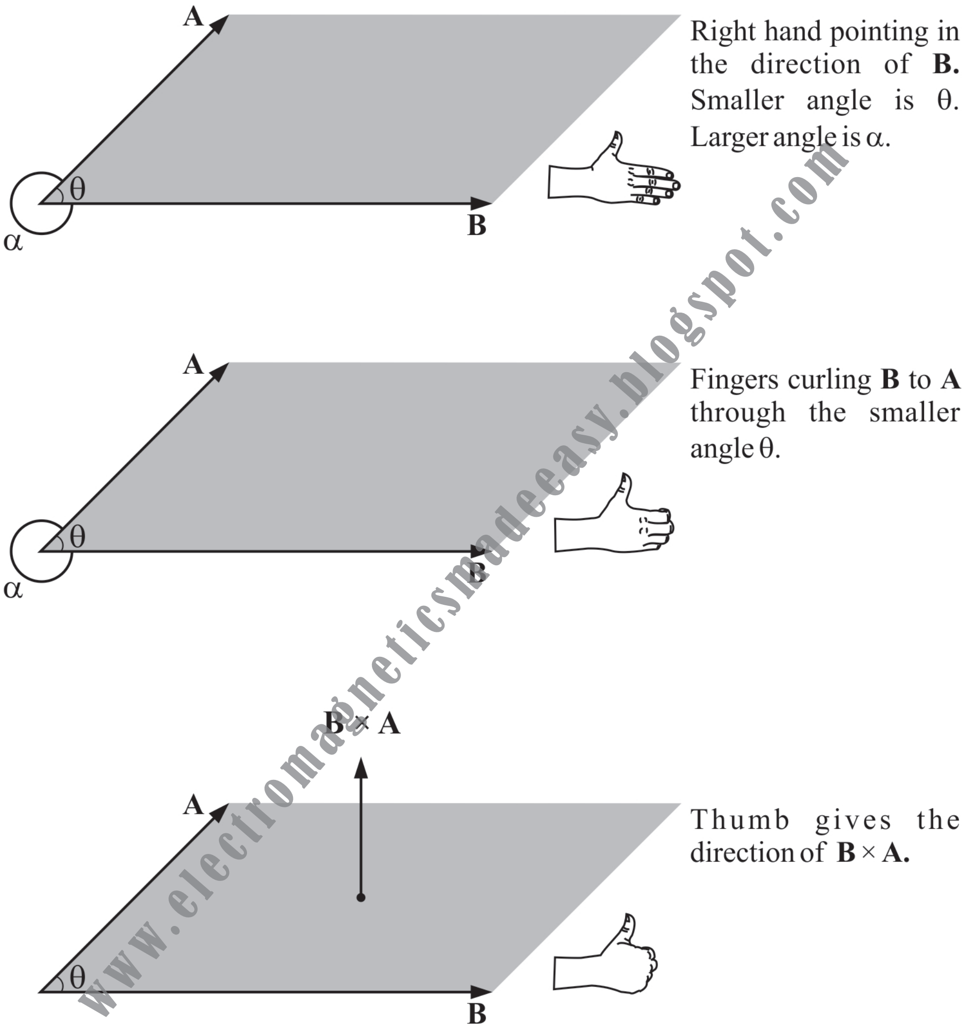

Fig.1.10b

1.10b. The direction of **A** × **B** and **B** × **A** as per right hand rule is indicated in figure 1.9. From the figure it is clear that

$$\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A}) \tag{1.8}$$

Hence the cross product is not commutative. However the cross product is distributive if the order of vectors is not changed.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}) \tag{1.9}$$

For two parallel vectors

$$\mathbf{A} \times \mathbf{B} = AB \sin 0 \;\; \hat{\mathbf{n}} = 1$$

The geometrical meaning of the cross product is, $|\mathbf{A} \times \mathbf{B}|$ is the area of the parallelogram generated by **A**, **B** in figure 1.9.

1.9 Vector Components and Unit Vectors

The above vector operations, which we discussed in previous sections does not involve any particular coordinate system. We will now rewrite the above operations in terms of components in the most widely used coordinate system – cartesian coordinate system. The unit vectors in the direction of *x, y, z* axis in the cartesian coordinate system is $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. We will discuss about vector components in two dimensions and then extend the results to three dimensions.

(a) Two Dimensions

Consider figure 1.11. The vector **A** shown in figure 1.11 can be resolved into components A_x, A_y. **A** can be written as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \tag{1.10}$$

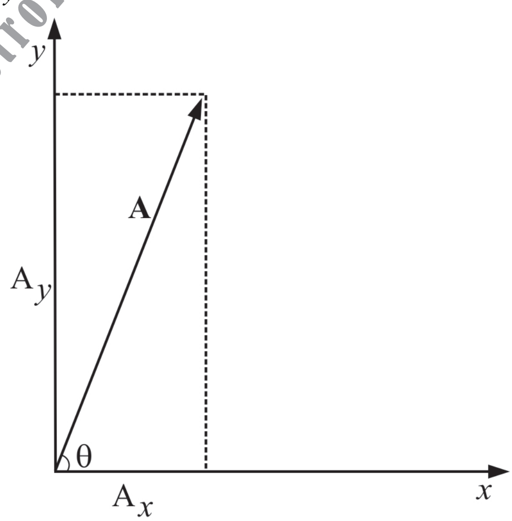

Fig.1.11

Also from figure 1.11

$$\cos\theta = \frac{A_x}{A}, \ \sin\theta = \frac{A_y}{A} \tag{1.11}$$

A the magnitude of **A** is

$$A = \sqrt{A_x^2 + A_y^2} \tag{1.12}$$

A_x, A_y are themselves scalars. $A_x\hat{\mathbf{i}}, A_y\hat{\mathbf{j}}$ are the component vectors. The **A** can point in any direction but the components vectors $A_x\hat{\mathbf{i}}, A_y\hat{\mathbf{j}}$ have constant direction along *x, y* axis respectively. In your lower classes you might have learnt that $\mathbf{A} = A\,\hat{\mathbf{A}}$ where $\hat{\mathbf{A}}$ is the unit vector in the direction of **A** and is given by

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A} = \frac{\mathbf{A}}{\sqrt{A_x^2 + A_y^2}} \tag{1.13}$$

$$\hat{\mathbf{A}} = \frac{A_x}{\sqrt{A_x^2 + A_y^2}}\,\hat{\mathbf{i}} + \frac{A_y}{\sqrt{A_x^2 + A_y^2}}\,\hat{\mathbf{j}} \tag{1.14}$$

To make the above points clearer let us consider an example. In figure 1.12a person A is pulling a block of iron along *x*-axis with a force of 3N using a thread tied to the block. The block will move along *x*-axis. In figure 1.12b person B is pulling the same block with a force of 4N along *y*-axis using a thread tied to the block. Now the block will move along *y*-axis. [In the above cases the numbers are only for indicative purpose]. Now consider figure 1.12c. Person A is pulling the block along *x* axis with a force of 3N while person B is pulling the block along *y* axis with a force of 4N using threads tied to the block. Assume that the threads are “invisible” – in the sense person A and person B are able to exert forces on the block (or) pull the block using the thread but the block is not constrained to move along *x* or *y* axis because of the thread or in other words the block can move in the *x-y* plane because of the forces acting on it without getting constrained by the threads.

Question: What is the total force $\mathbf{F}_t$ acting on the block. What is the direction of $\mathbf{F}_t$ or in which direction the block will move?

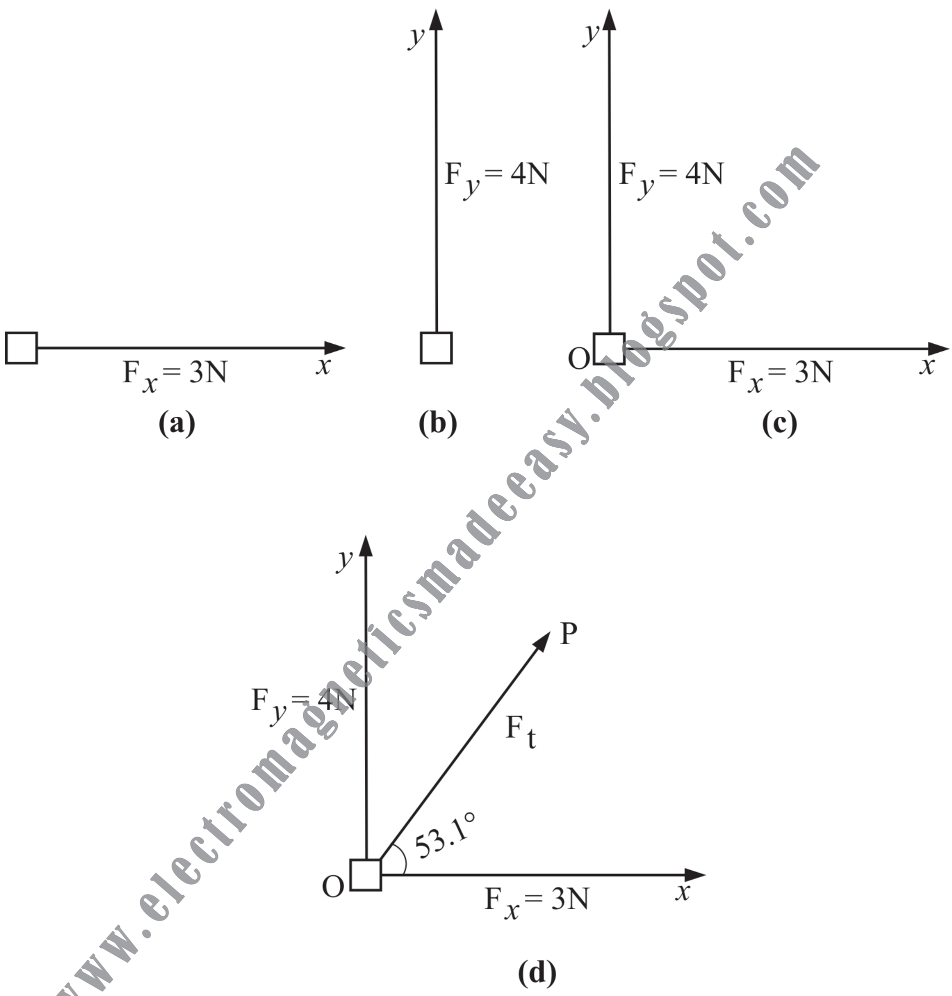

Fig.1.12

The total force $\mathbf{F}_t$ acting on the block is

$$\mathbf{F}_t = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$$

The magnitude of the total force is

$$F_t = \sqrt{9+16} = 5\text{N}$$

The angle made by the total force with x axis is

$$\cos\theta = \frac{F_x}{F_t} = \frac{3}{5}$$

$$\Rightarrow \theta = 53.1°$$

The angle is made by F_t with y axis is

$$= 90 - 53.1 = 36.9°$$

Thus in figure 1.12d a total force of 5N acts on the block along the line OP, where the line OP makes an angle of 53.1° with the x-axis.

Let us restate the above problem in a slightly different manner. In fig.1.13 person C is pulling the block along OP with an invisible thread, with a force of $F_t = 5\text{N}$

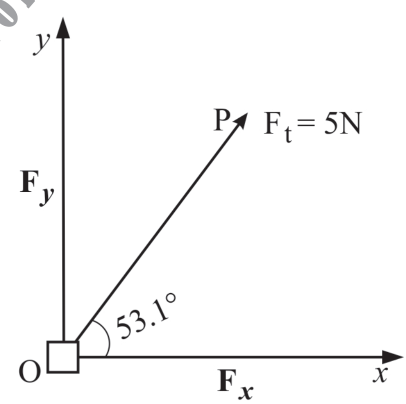

Fig.1.13

The line OP makes an angle of 53.1° with the *x*-axis.

Question: Resolve the force $\mathbf{F}_t$ into its *x* and *y* components.

x component F_x is given by

$$F_x = F_t \cos\theta = 5 \cos 53.1 = 3N$$

y component F_y is given by

$$F_y = F_t \sin\theta = 5 \sin 53.1$$

$$F_y = 4N.$$

Thus person C pulling the block along line OP in figure 1.13 is equivalent to person A and B pulling the same block with a force of 3N and 4N along *x* and *y* axis respectively as in figure 1.12d. Or in other words, we have resolved F_t in figure 1.13 into equivalent *x* and *y* components F_x, F_y respectively. The same is true for any vector. That is, any vector can be resolved into its components.

What about resolving scalars. To find out, consider a particle of mass 5 kg located as shown in figure 1.14. Can you resolve the mass of 5 kg into its components

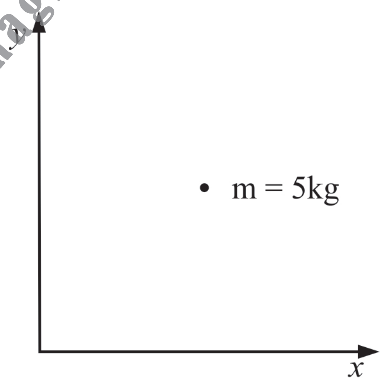

Fig.1.14

along *x* and *y* axis. Mass being a scalar quantity is independent of direction and cannot be resolved into components. The same is true for any scalar.

(b) Three Dimensions

Now let us generalize the above results for three dimensions. In figure 1.15a we show *x, y, z* along with their unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. The vector **A** shown in figure 1.15b can be written in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{1.15}$$

and

$$\cos\alpha = \frac{A_x}{A},\ \cos\beta = \frac{A_y}{A},\ \cos\Gamma = \frac{A_z}{A} \tag{1.16}$$

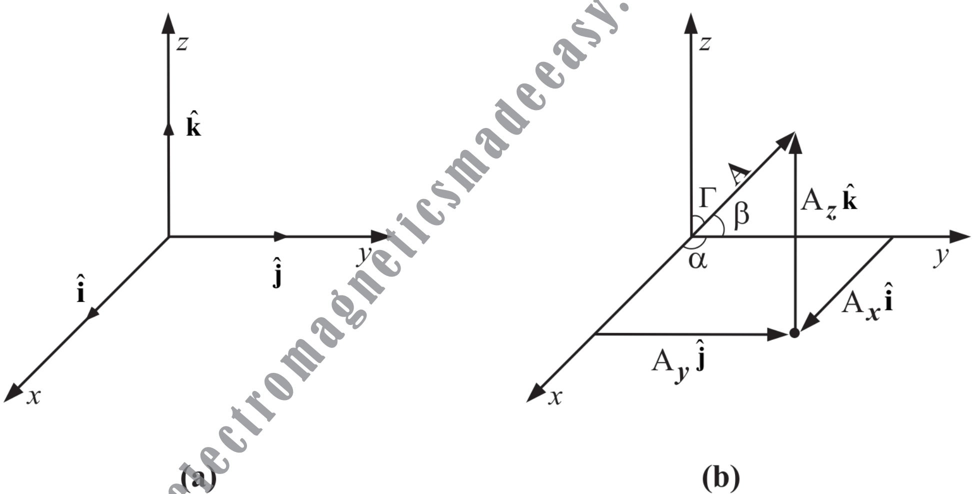

Fig.1.15

A the magnitude of **A** is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.17}$$

A_x, A_y, A_z are themselves scalars. $A_x\hat{\mathbf{i}}, A_y\hat{\mathbf{j}}, A_z\hat{\mathbf{k}}$ are the component vectors. The unit vector $\hat{\mathbf{A}}$ is given by

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A} = \frac{A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{1.18}$$

$$\hat{\mathbf{A}} = \frac{A_x}{\sqrt{A_x^2+A_y^2+A_z^2}}\,\hat{\mathbf{i}} + \frac{A_y}{\sqrt{A_x^2+A_y^2+A_z^2}}\,\hat{\mathbf{j}} + \frac{A_z}{\sqrt{A_x^2+A_y^2+A_z^2}}\,\hat{\mathbf{k}} \qquad (1.19)$$

Writing any vector in terms of its components has number of advantages.

(c) Vector addition and subtraction in terms of components:

To add any two vectors **A**, **B** just add the components

$$\mathbf{A}+\mathbf{B}=(A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}})+(B_x\,\hat{\mathbf{i}}+B_y\,\hat{\mathbf{j}}+B_z\,\hat{\mathbf{k}})$$

$$\mathbf{A}+\mathbf{B}=(A_x+B_x)\hat{\mathbf{i}}+(A_y+B_y)\hat{\mathbf{j}}+(A_z+B_z)\hat{\mathbf{k}} \qquad (1.20)$$

To subtract two vectors **A, B** just subtract the components

$$\mathbf{A}-\mathbf{B}=(A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}})-(B_x\,\hat{\mathbf{i}}+B_y\,\hat{\mathbf{j}}+B_z\,\hat{\mathbf{k}})$$

$$\mathbf{A}-\mathbf{B}=(A_x-B_x)\hat{\mathbf{i}}+(A_y-B_y)\hat{\mathbf{j}}+(A_z-B_z)\hat{\mathbf{k}} \qquad (1.21)$$

Addition, subtraction of components is similar to that of scalars.

(d) Multiplication of a vector by a scalar:

To multiply a vector by a scalar say u multiply each component by a respective scalar

$$u\,\mathbf{A}=u\,(A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}})$$

$$u\,\mathbf{A}=u\,A_x\,\hat{\mathbf{i}}+u\,A_y\,\hat{\mathbf{j}}+u\,A_z\,\hat{\mathbf{k}} \qquad (1.22)$$

(e) Dot product in terms of components:

$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are mutually perpendicular to each other. Hence

$$\hat{\mathbf{i}}.\,\hat{\mathbf{i}}=\hat{\mathbf{j}}.\hat{\mathbf{j}}=\hat{\mathbf{k}}.\hat{\mathbf{k}}=1$$

$$\hat{\mathbf{i}}.\,\hat{\mathbf{j}}=\hat{\mathbf{i}}.\hat{\mathbf{k}}=\hat{\mathbf{j}}.\hat{\mathbf{k}}=0 \qquad (1.23)$$

Thus the dot product can be written as

$$\mathbf{A}.\mathbf{B}=(A_x\,\hat{\mathbf{i}}+A_y\,\hat{\mathbf{j}}+A_z\,\hat{\mathbf{k}}).(B_x\,\hat{\mathbf{i}}+B_y\,\hat{\mathbf{j}}+B_z\,\hat{\mathbf{k}})$$

$$\mathbf{A}.\mathbf{B}=A_x\,B_x+A_y\,B_y+A_z\,B_z \qquad (1.24)$$

(f) Cross product in terms of components:

$$\left.\begin{aligned}&\hat{\mathbf{i}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}\times\hat{\mathbf{k}}=0\\&\hat{\mathbf{i}}\times\hat{\mathbf{j}}=-\hat{\mathbf{j}}\times\hat{\mathbf{i}}=\hat{\mathbf{k}}\\&\hat{\mathbf{j}}\times\hat{\mathbf{k}}=-\hat{\mathbf{k}}\times\hat{\mathbf{j}}=\hat{\mathbf{i}}\\&\hat{\mathbf{k}}\times\hat{\mathbf{i}}=-\hat{\mathbf{i}}\times\hat{\mathbf{k}}=\hat{\mathbf{j}}\end{aligned}\right\} \tag{1.25}$$

Therefore

$$\mathbf{A}\times\mathbf{B}=(A_x\hat{\mathbf{i}}+A_y\hat{\mathbf{j}}+A_z\hat{\mathbf{k}})\times(B_x\hat{\mathbf{i}}+B_y\hat{\mathbf{j}}+B_z\hat{\mathbf{k}}) \tag{1.26}$$

$$=(A_yB_z-A_zB_y)\hat{\mathbf{i}}+(A_zB_x-A_xB_z)\hat{\mathbf{j}}+(A_xB_y-A_yB_x)\hat{\mathbf{k}} \tag{1.27}$$

Which can be written in much more familiar from as

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} \tag{1.28}$$

1.10 Triple Products

Scalar triple product is defined as **A** . (**B** × **C**). The geometrical meaning of **A**. (**B** × **C**) can be understood with the aid of figure 1.16.

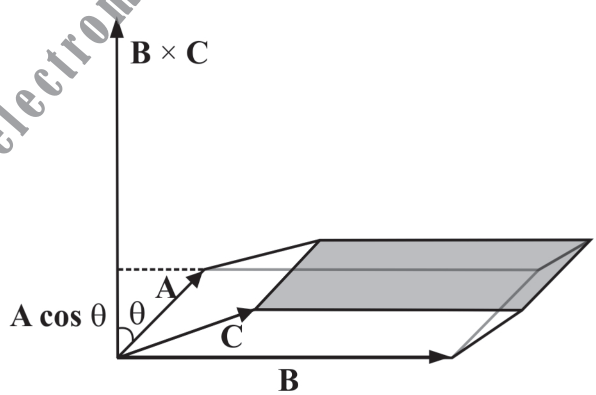

Fig.1.16

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \mathrm{A}\cos\theta(\mathbf{B}\times\mathbf{C}) \tag{1.29}$$

We have already seen that cross product $\mathbf{B} \times \mathbf{C}$ is the area of the parallelogram [Area of the base in the figure 1.16]. Also from figure 1.16 $\mathrm{A}\cos\theta$ is the vertical height of the parallelepiped. Hence,

$\mathbf{A} . (\mathbf{B} \times \mathbf{C})$ = (Vertical height of parallelepiped) × (Area of base)

= Volume of the parallelepiped.

Thus $\mathbf{A} . (\mathbf{B} \times \mathbf{C})$ is the volume of the parallelepiped, whose sides are **A**, **B** and **C**. As any face of the parallelepiped can be taken as the base

$\mathbf{A} . (\mathbf{B} \times \mathbf{C}) = \mathbf{B} . (\mathbf{C} \times \mathbf{A}) = \mathbf{C} . (\mathbf{A} \times \mathbf{B})$

Note that in the above expression cyclic alphabetical order **A**, **B**, **C** is maintained. In component form

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathrm{A}_x & \mathrm{A}_y & \mathrm{A}_z \\ \mathrm{B}_x & \mathrm{B}_y & \mathrm{B}_z \\ \mathrm{C}_x & \mathrm{C}_y & \mathrm{C}_z \end{vmatrix} \tag{1.30}$$

(b) Vector triple product:

The vector triple product is defined as

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} . \mathbf{C}) - \mathbf{C}(\mathbf{A} . \mathbf{B}) \tag{1.31}$$

Kindly note we have written $\mathbf{B}(\mathbf{A} . \mathbf{C})$. The statement $\mathbf{B} . (\mathbf{A} . \mathbf{C})$ has no meaning. Similarly the statement $\mathbf{B} \times (\mathbf{A} . \mathbf{C})$ also has no meaning.

Example 1.1

Find the value of t if the vectors $\mathbf{A} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{B} = 5\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ are perpendicular to each other.

Solution:

If **A** and **B** are perpendicular to each other then $\mathbf{A} . \mathbf{B} = 0$

$$\Rightarrow (2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}).(5\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = 0$$
$$\Rightarrow 10 + t + 6 = 0$$
$$\Rightarrow t = -16$$

Vector Analysis

Example 1.2

Calculate the angle between the vectors $\mathbf{A} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \mathbf{B} = 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$

Solution:

We have

$$\mathbf{A}.\mathbf{B} = \left(3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}\right).\left(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 9\hat{\mathbf{k}}\right)$$

$$= 9 - 6 + 18 = 21.$$

$$\mathrm{A} = \sqrt{9+1+4} = \sqrt{14}$$

$$\mathrm{B} = \sqrt{9+36+81} = \sqrt{126}$$

We know that

$$\mathbf{A}.\mathbf{B} = \mathrm{A\,B}\cos\theta$$

$$\cos\theta = \frac{\mathbf{A}.\mathbf{B}}{\mathrm{AB}} = \frac{21}{\sqrt{14}\sqrt{126}}$$

$$\theta = \cos^{-1}\left(\frac{21}{\sqrt{14}\ \sqrt{126}}\right)$$

$$\theta = 60°$$

Example 1.3

Check whether the vectors $\mathbf{A} = 6\hat{\mathbf{i}} - 9\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{B} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ are parallel to each other.

Solution:

We have

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & -9 & -3 \\ -2 & 3 & 1 \end{vmatrix}$$

$$= \hat{\mathbf{i}}[-9+9]-\hat{\mathbf{j}}[6-6]+\hat{\mathbf{k}}[18-18] = 0$$
$$= 0$$

$\mathbf{A} \times \mathbf{B} = 0$

$$\Rightarrow \text{A B } \sin\theta = 0$$

As A, B the magnitude of **A**, **B** are not zero

$\sin\theta = 0 \;\; \Rightarrow \theta = 0.$

Hence **A**, **B** are parallel to each other.

Example 1.4

Check whether the vectors $\mathbf{A}=2\hat{\mathbf{i}}+4\hat{\mathbf{j}}+18\hat{\mathbf{k}}, \mathbf{B}=9\hat{\mathbf{i}}+3\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ and $\mathbf{C}=4\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ are coplanar.

Solution

The condition that the three vectors are coplanar is that their scalar triple product is zero

$\mathbf{A} \, . \, (\mathbf{B} \times \mathbf{C}) = 0$

we have.

$$\mathbf{A} \, . \, (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 2 & 4 & 18 \\ 9 & 3 & 6 \\ 4 & 2 & 6 \end{vmatrix}$$

$$= 2\,(18–12) – 4\,(54–24) + 18\,(18–12)$$

$$= 2\,(6) – 4\,(30) + 18\,(16)$$

$$= 0.$$

Hence the vectors are coplanar.

1.11 Line, Surface and Volume Integration

In this section we will learn about the integration of vectors. There are three different types of vector integration namely line, surface and volume integrations.

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Vector Analysis

(a) Line Integration

Let us assume a scalar function u and a vector function **A**. There are three important line integration

$$\int_P^Q u\,d\boldsymbol{l},\ \int_P^Q \mathbf{A}.d\boldsymbol{l},\ \int_P^Q \mathbf{A}\times d\boldsymbol{l},$$

where in general the line integral will depend on the path. For example the above integrations will yield different results for path 1, 2 shown in figure 1.17.

Out of the above three integrals $\int_P^Q \mathbf{A}.d\boldsymbol{l}$ will often appear in electromagnetics.

If the path over which the line integral is carried out is closed we shall put a circle over the integration indicating that the path is closed

$$\oint \mathbf{A}.d\boldsymbol{l}$$

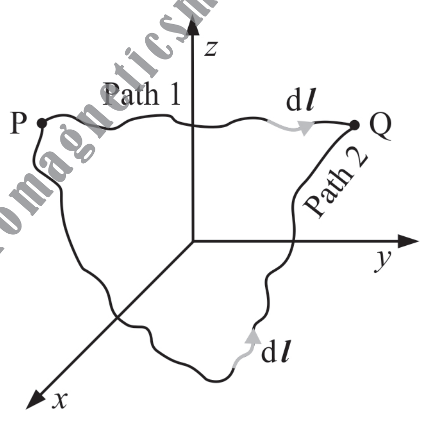


Fig.1.17

Although in general the line integration depends on path, for certain vectors the line integration doesn't depend on path, but only on end points. For example for these certain vectors the line integration over the path 1 and path 2 in figure 1.17 will yield same result and will depend on only the end points P,Q.

(b) Surface Integration

Consider a surface **S**. Let the surface **S** be divided into small elements d**s** as shown in figure 1.18. There are three different surface integrals.

$$\iint_S u\, d\mathbf{s},\ \iint_S \mathbf{A}.d\mathbf{s},\ \iint_S \mathbf{A}\times d\mathbf{s}.$$

where u is a scalar. Out of the three integrals the surface integral $\iint_S \mathbf{A}.d\mathbf{s}$

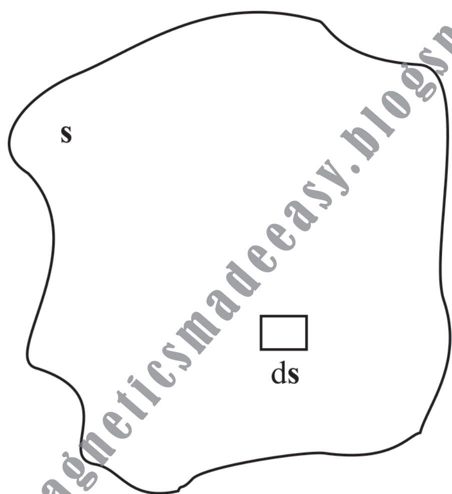


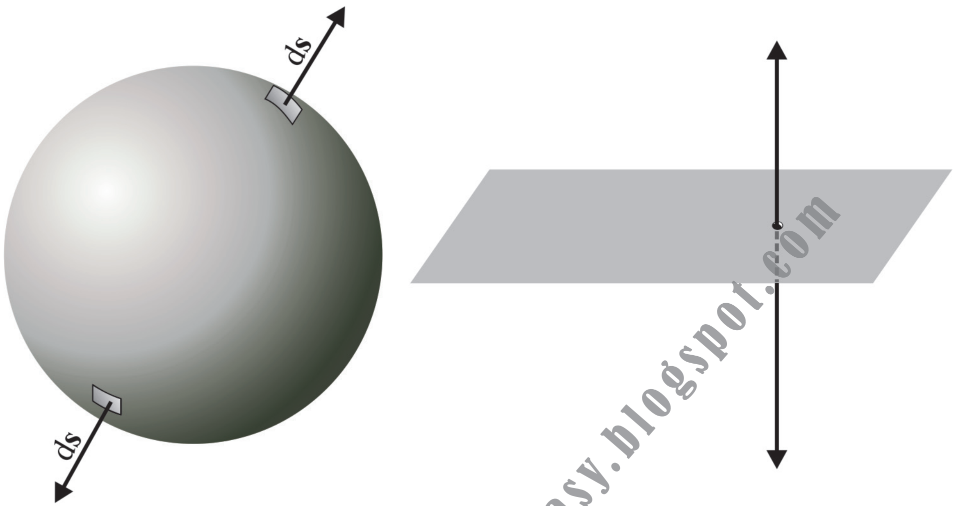
Fig.1.18

will often appear in electromagnetics and it is customary to write the above integral for open and closed surface as

$$\int_S \mathbf{A}.d\mathbf{s} \text{ and } \oint_S \mathbf{A}.d\mathbf{s}$$

Surface is a vector and has direction. For a closed surface [assume the surface of a tennis ball, which is a closed surface] the outward drawn normal is the direction of the surface at that point as shown in figure 1.19a.

For a open surface the direction of the surface is the perpendicular direction to the surface at that point [Assume a piece of paper which is a open surface]. However there are two perpendicular direction to the open surface and hence the direction is arbitrary as shown in figure 1.19b.



(a) (b)

Fig.1.19

(c) Volume Integration

Volume is a scalar and hence performing volume integration is easy. There are two different volume integration

$$\iiint_\tau u\, d\tau, \iiint_\tau \mathbf{A}\, d\tau$$

or

$$\int_\tau u\, d\tau, \int_\tau \mathbf{A}\, d\tau$$

where τ is the volume over which the integration is performed and dτ is the volume element.

1.12 Flux

Flux is a important quantity which we will encounter often in electromagnetics. In this section we will learn how to define flux.

Assume a rectangular cardboard abcd as shown in figure 1.20, whose surface area is S. Suppose the cardboard is exposed to sun rays and is held perpendicular to the sun rays as shown in figure 1.20a. A shadow forms below as shown in the figure. Because the cardboard is held perpendicular to the sun rays the area of the shadow formed will be exactly S.

However now if the cardboard is held at an angle θ to the sun rays that is, instead of being perpendicular to the sun rays, the cardboard is inclined with respect to the direction of the sun rays at an angle θ as shown in figure 1.20b.

Now the area of the shadow will be lesser. This is because the effective area shown by the card board to the incoming sun rays is lesser than S, when the card board is inclined at an angle θ to the direction of sun rays. When the card board is perpendicular to the sun rays all the area S of the card board is shown to the

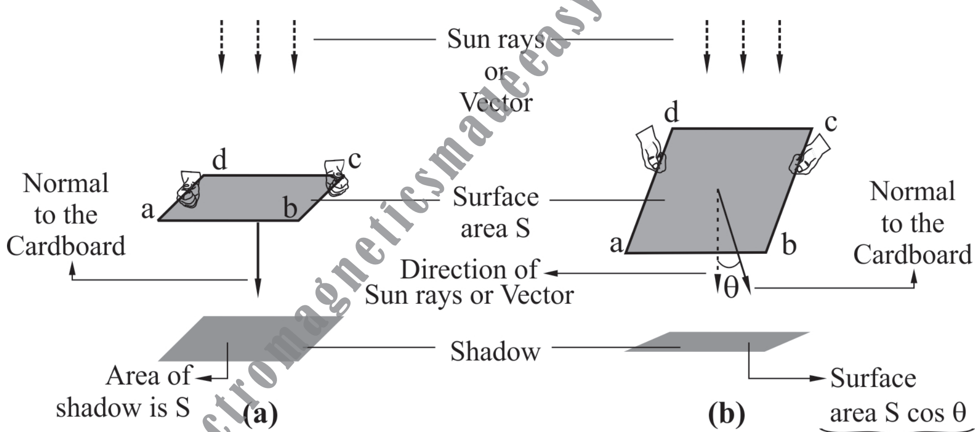


Fig.1.20

sun rays. Hence the area of the shadow formed is S. But when the card board is inclined at angle θ to the sun rays the effective area of the card board shown to the sun rays is $S\cos\theta$ and hence the area of the shadow formed is $S\cos\theta$. In other words the projection of surface S is $S\cos\theta$. Of course when the card board is held perpendicular to the sun rays $\theta=0$ and hence the effective area is $S\cos 0 = S$.

Now suppose in the place of card board assume a surface abcd whose area is S and in the place of sun rays assume an electric field **E** vector. The effective area shown by the surface abcd to the incoming vector **E** will be $S\cos\theta$.

Vector Analysis

Flux is defined as the number of lines passing through the given area. Hence the total flux passing through the surface abcd in figure 1.20b is

$$= \mathbf{E}(S\cos\theta) = \mathbf{E}.\mathbf{S}$$

However if we divide the surface into small elements d**s** then the total flux through abcd will be

$$\text{Flux} = \iint_s \mathbf{E}.\,d\mathbf{s}$$

In section 1.11b we have treated area as a vector. Throughout electromagnetics we will be treating area as a vector. Now a general question is whether area is vector or a scalar. In section 1.1 and in figure 1.1 masses sum up and give the same result in whichever direction they are added. Quantities which behave like mass are called scalars. In figure 1.2 forces sum up or cancel out depending upon the direction in which they act. Quantities which behave like force are called vectors. If you add area of two plots (empty land) they sum up irrespective of whatever direction they are added and hence area must be a scalar. But in electromagnetics we treat area as vector. How is this possible? The answer lies in figure 1.20. The effective area shown by area abcd to the incoming vector depends on the "DIRECTION" of orientation of normal of the area abcd to the incoming vector **E**. If the "DIRECTION" of the normal of the area abcd is along the direction of the vector then the effective area shown to the incoming vector is $S\cos 0 = S$. If the "DIRECTION" of the normal of the area abcd is oriented at an angle θ to the incoming vector the effective area shown to the incoming vector is $S\cos\theta$.

"Because the amount of effective area shown by the area abcd to the incoming vector depends on the angle made by 'DIRECTION' of the normal of the area abcd to the incoming vector, area must be treated as vector".

1.13 Vector Differentiation: Gradient of a Scalar Function

There are three important types of vector differentiation-Gradient, divergence and curl. In this section we will discuss about gradient. In sections 1.14, 1.15 we will discuss about divergence and curl.

As we will see the gradient of a scalar function u in Cartesian coordinates is given by

$$\nabla \mathrm{u} = \frac{\partial \mathrm{u}}{\partial x}\hat{\mathbf{i}} + \frac{\partial \mathrm{u}}{\partial y}\hat{\mathbf{j}} + \frac{\partial \mathrm{u}}{\partial z}\hat{\mathbf{k}}$$

The physical meaning of gradient will be clear to the reader if the reader considers $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives the rate of change of y with respect to x when y is a function of x alone $y = y(x)$. However we have number of scalar functions which depend on three variables (x, y, z) say for example $\mathrm{u} = \mathrm{u}(x, y, z)$. Similar to $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\nabla \mathrm{u}$ gives the maximum rate of change of u with respect to x, y, z. As we have $\frac{\mathrm{d}y}{\mathrm{d}x}$ when y depends on single variable $y = y(x)$, we have $\nabla \mathrm{u}$ when the given scalar function u depends on three variables (x, y, z).

Consider a scalar function $\mathrm{u}(x, y, z)$ which is constant over the surface 1 as shown in figure 1.21. Consider another surface, surface 2 on which the scalar function has a value u + du where du is small change in u. Point A lies on surface 1 and points B, C lies on surface 2. Consider an arbitrary origin and with respect to the origin let the position vectors of A and C be **r** and **r** + d**r**. We have AC = d**r.**

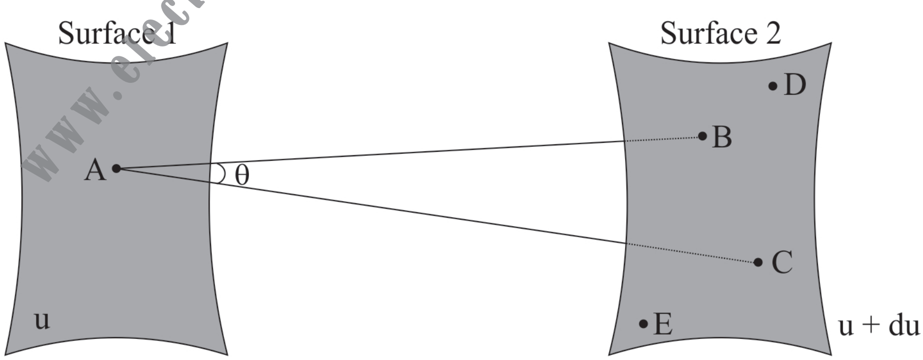


Fig.1.21

In Cartesian coordinates

$$\mathrm{d}\mathbf{r} = \mathrm{d}x\,\hat{\mathbf{i}} + \mathrm{d}y\,\hat{\mathbf{j}} + \mathrm{d}z\,\hat{\mathbf{k}} \tag{1.32}$$

From calculus

$$\mathrm{du} = \frac{\partial \mathrm{u}}{\partial x}\mathrm{d}x + \frac{\partial \mathrm{u}}{\partial y}\mathrm{d}y + \frac{\partial \mathrm{u}}{\partial z}\mathrm{d}z \tag{1.33}$$

The above expression can be written as

$$\mathrm{du} = \left(\hat{\mathbf{i}}\frac{\partial \mathrm{u}}{\partial x} + \hat{\mathbf{j}}\frac{\partial \mathrm{u}}{\partial y} + \hat{\mathbf{k}}\frac{\partial \mathrm{u}}{\partial z}\right).\left(\mathrm{d}x\,\hat{\mathbf{i}} + \mathrm{d}y\,\hat{\mathbf{j}} + \mathrm{d}z\,\hat{\mathbf{k}}\right) \tag{1.34}$$

Let us denote the vector $\left(\hat{\mathbf{i}}\frac{\partial \mathrm{u}}{\partial x} + \hat{\mathbf{j}}\frac{\partial \mathrm{u}}{\partial y} + \hat{\mathbf{k}}\frac{\partial \mathrm{u}}{\partial x}\right)$ as $\mathbf{G}$.

Hence du = **G**. d**r** (1.35)

Now let us assume point C lies an surface 1. In that case du = 0.

Hence equation 1.35 is

G . d**r** = 0.

G dr cos α = 0 (1.36)

where α is the angle between **G** and d**r**. We will first find direction of **G**. Equation 1.36 can be zero only if α = 90°. As d**r** points from A to C, d**r** will be lying on surface 1 when point C lies on surface 1. As α the angle between d**r** and **G** is 90° and d**r** is lying on surface 1, the vector **G** is normal to the surface 1. We have identified the direction of **G**. It is normal to the surface 1. In figure 1.21 let **AB** = d**n**. From figure 1.21

$$\cos\theta = \frac{\mathrm{d}n}{\mathrm{d}r} \tag{1.37}$$

$$\Rightarrow \mathrm{d}n = \mathrm{d}r\cos\theta \tag{1.38}$$

Let $\hat{\mathbf{n}}$ be the unit vector in the direction of **AB**. Then equation 1.38 can be written as

$$\mathrm{d}n = \hat{\mathbf{n}}\ .\ \mathrm{d}\mathbf{r} \tag{1.39}$$

As we move from point A to B u increases by du. Hence

u = u (n)

$$du = \frac{\partial u}{\partial n} dn \tag{1.40}$$

Substituting (1.39) in (1.40)

$$du = \frac{\partial u}{\partial n} \hat{\mathbf{n}} . d\mathbf{r} \tag{1.41}$$

Comparing (1.41) and (1.35)

$$\mathbf{G} = \frac{\partial u}{\partial n} \hat{\mathbf{n}} \tag{1.42}$$

Thus the magnitude of **G** is $\frac{\partial u}{\partial n}$ and its direction is normal to the surface 1. Consider $\frac{\partial u}{\partial n}, \frac{\partial u}{\partial r}$ in figure 1.21. We see that out of the two derivatives $\frac{\partial u}{\partial n}, \frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial n}$ is greater than $\frac{\partial u}{\partial r}$ because dn in figure 1.21 is lesser than dr. If we consider D, E or any other point on surface 2 always AB = dn will be minimum as compared to AD, AE etc. Hence $\frac{\partial u}{\partial n}$ will be the maximum value of the derivative pointing normal to surface 1.

Hence from equations 1.34, 1.35, 1.42

$$\mathbf{G} = \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}} = \frac{\partial u}{\partial n}\hat{\mathbf{n}} \tag{1.43}$$

"Thus the gradient of the scalar function u is a vector. Its magnitude is equal to the maximum rate of change of scalar function u with respect to scalar variables. The direction of this vector is along this maximum rate of change of scalar function"

We denote the gradient of a scalar function as grad u or ∇u.

$$\mathbf{G} = \text{grad}\, u = \nabla u = \hat{\mathbf{i}}\frac{\partial u}{\partial x} + \hat{\mathbf{j}}\frac{\partial u}{\partial y} + \hat{\mathbf{k}}\frac{\partial u}{\partial z} \tag{1.44}$$

The above equation can be written as

$$\nabla \mathrm{u} = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) \mathrm{u} \tag{1.45}$$

Here $\nabla = \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}$ is called " del" operator.

$\nabla \mathrm{u}$ serves the same purpose in three dimensions as $\frac{\mathrm{d}y}{\mathrm{d}x}$ in one dimension.

That is in figure 1.21. if we represent **AD** = d$\mathbf{r}_1$**, AE** = d$\mathbf{r}_2$, etc. We have number of derivations $\frac{\partial \mathrm{u}}{\partial \mathrm{n}}, \frac{\partial \mathrm{u}}{\partial \mathrm{r}}, \frac{\partial \mathrm{u}}{\partial r_1}, \frac{\partial \mathrm{u}}{\partial \mathrm{r}_2} \cdots$ etc. Out of the above available derivatives $\frac{\partial \mathrm{u}}{\partial \mathrm{n}}$ is the derivative selected to represent the rate change of the scalar function with space variables (*x, y, z*) because $\frac{\partial \mathrm{u}}{\partial \mathrm{n}}$ gives the maximum rate of change of the scalar function with respect to space variables.

As $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives the rate of change of *y* with respect to *x* when *y* depends on *x* only that is $y = y(x)$, $\nabla \mathrm{u}$ gives the information how fast u varies with *x, y, z* when u depends on (*x, y, z*) that is u = u (*x, y, z*). Some of the examples in which the given scalar function depends on three variables (*x, y, z*) are pressure of a gas in a cubical box P = P (*x, y, z*), temperature inside a room T (*x, y, z*).

1.14 Vector Differentiation: Divergence of a Vector

The word divergence reminds us the word "diverge" which means to "spread out". Divergence is the measure of how much the given vector diverges or spreads out. For example for the vector shown in figure 1.22a the vector spreads out and is having large positive divergence. The vector shown in figure 1.22b is having large negative divergence. The vector shown in figure 1.22c is having zero divergence.

Suppose we put a hole (marked p in figure 1.23a) in a floor and fix a pipe in it and allow water to come out as shown in Figure 1.23a. Water spreads out; hence water is diverging and is having positive divergence [in vectorial terms the velocity

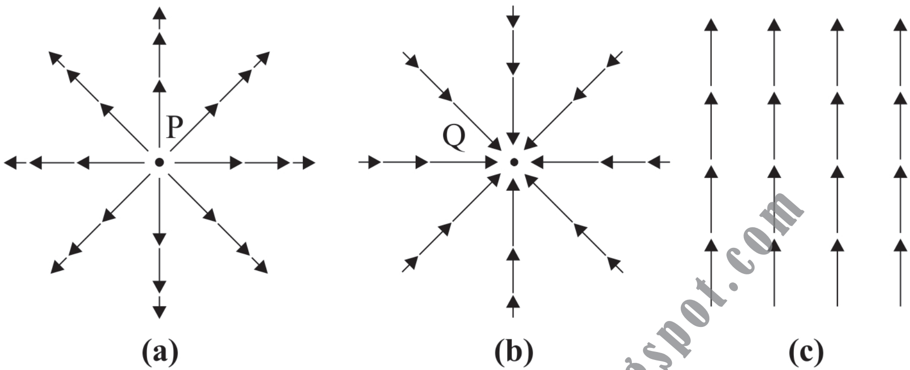


Fig.1.22

of water is diverging]. On the other hand, we put a hole (marked q in figure 1.23b) in a cone like structure and allowed water to flow inside, from the edges of the cone, then the water is having negative divergence [in vectorial terms it is the velocity of water having negative divergence].

We can get another important physical information from figure 1.23. In figure 1.23a point p is acting like a source from which water emanates from and diverges. Hence presence of positive divergence implies implicitly that there is a

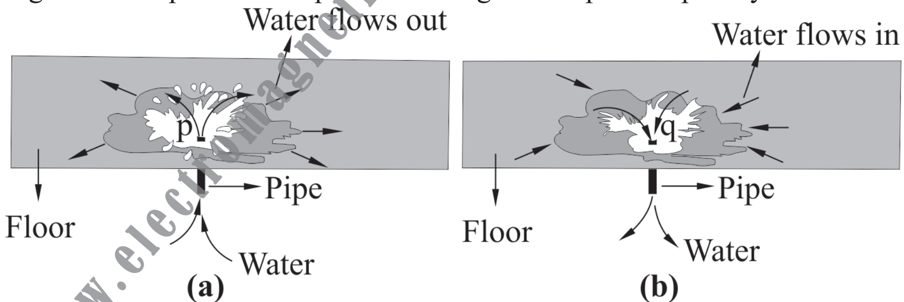


Fig.1.23

source for the associated vector from which the vector emanates from. In figure 1.23b point q is acting like a sink into which water flows in. Hence presence of negative divergence implies that there is a sink for the associated vector to which the vector converges to.

Vector Analysis

In Cartesian coordinates the divergence is defined as [for any vector **A**]

$$\nabla . \mathbf{A} = \left[\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right] . \left[\hat{\mathbf{i}}\,\mathrm{A}_x + \hat{\mathbf{j}}\,\mathrm{A}_y + \hat{\mathbf{k}}\,\mathrm{A}_z\right] \qquad (1.46)$$

$$\nabla . \mathbf{A} = \frac{\partial \mathrm{A}_x}{\partial x} + \frac{\partial \mathrm{A}_y}{\partial y} + \frac{\partial \mathrm{A}_z}{\partial z} \qquad (1.47)$$

1) In figure 1.22a if the diverging vector is **A** then

$\nabla . \mathbf{A} > 0$ [This implies that there is a source for **A**. Point P in figure 1.22a is acting like a source for **A**].

2) In figure 1.22b if the vector depicted is **A** having negative divergence then

$\nabla . \mathbf{A} < 0$ [This implies that there is a sink for **A**. Point Q in figure 1.22b is acting like a sink for **A**]

3) In figure 1.22c if the depicted vector is **A** then

$\nabla . \mathbf{A} = 0$ [There is no source or sink for **A**]

As we will see in chapter 4 the above example will be used to explain magnetic monopoles do not exist.

A vector alone can have divergence and a scalar cannot have divergence. For example if the mass of an object is 200kg, it is 200kg as itself and it cannot diverge or spread out. [If the mass of 200kg explodes and spreads out, it will be the velocity of each piece which will be diverging, not the mass of the individual piece].

We will now see the mathematical description of divergence. As shown in figure 1.22 the divergence of any vector **A** represents flux. In figure 1.22a there is outward flow of flux from the source at point P. In figure 1.22 b there is inward flow of flux to the sink at point Q. In figure 1.22c there is an equal amount of inward and outward flux going through any volume resulting in zero net flux.

"Thus the divergence of a vector function **A** is defined as the flux per unit volume in the limit of volume enclosed by the closed surface tends to zero".

Consider a point P' whose coordinates are x', y', z' with respect to some arbitrary origin. Consider a small closed surface S surrounding the point P'. Let τ be the small volume enclosed by the surface S. Then the divergence of **A** at point P' is defined as

$$\operatorname{div}\mathbf{A} = \lim_{\tau \to 0} \frac{\oint \mathbf{A}.\mathbf{ds}}{\tau} \tag{1.48}$$

Where in the above equation $\oint \mathbf{A}.\mathbf{ds}$ represents flux [see section 1.12]

Consider a infinitesimal rectangular box whose sides are $\Delta x, \Delta y, \Delta z$ as shown in figure 1.24. The box has six face ABQP, CRSD, ABCD, PQRS, BCRQ, ADSP. From figure 1.24

Area $(\Delta \mathbf{s})_{\mathrm{ABQP}} = \Delta y \Delta z\, \hat{\mathbf{i}}$

Area $(\Delta \mathbf{s})_{\mathrm{CRSD}} = -\Delta y \Delta z\, \hat{\mathbf{i}}$

Area $(\Delta \mathbf{s})_{\mathrm{ABCD}} = -\Delta x \Delta z\, \hat{\mathbf{j}}$

Area $(\Delta \mathbf{s})_{\mathrm{PQRS}} = \Delta x \Delta z\, \hat{\mathbf{j}}$

Area $(\Delta \mathbf{s})_{\mathrm{BCRQ}} = \Delta x \Delta y\, \hat{\mathbf{k}}$

Area $(\Delta \mathbf{s})_{\mathrm{ADSP}} = -\Delta x \Delta y\, \hat{\mathbf{k}}$

The vector **A** in Cartesian coordinates can be written as

$$\mathbf{A} = \mathrm{A}_x \hat{\mathbf{i}} + \mathrm{A}_y \hat{\mathbf{j}} + \mathrm{A}_z \hat{\mathbf{k}}$$

For the box shown in figure 1.24 the integral $\oint \mathbf{A}.\mathbf{ds}$ can be written as

$$\oint_{\mathrm{S}} \mathbf{A}.\mathbf{ds} = \int_{\mathrm{ABQP}} \mathbf{A}.\mathbf{ds} + \int_{\mathrm{CRSD}} \mathbf{A}.\mathbf{ds} + \int_{\mathrm{ABCD}} \mathbf{A}.\mathbf{ds}$$

$$+ \int_{\mathrm{PQRS}} \mathbf{A}.\mathbf{ds} + \int_{\mathrm{BCRQ}} \mathbf{A}.\mathbf{ds} + \int_{\mathrm{ADSP}} \mathbf{A}.\mathbf{ds} \tag{1.49}$$

Now consider the integral $\int_{\mathrm{ABQP}} \mathbf{A}.\mathbf{ds}$

$$\int_{\mathrm{ABQP}} \mathbf{A}.\mathbf{ds} = (\mathbf{A})_{\mathrm{ABQP}}.(\Delta \mathbf{s})_{\mathrm{ABQP}} \tag{1.50}$$

$$= \left[\mathrm{A}_x \hat{\mathbf{i}} + \mathrm{A}_y \hat{\mathbf{j}} + \mathrm{A}_z \hat{\mathbf{k}}\right]_{\mathrm{ABQP}}.\left(\Delta y \Delta z\, \hat{\mathbf{i}}\right) \tag{1.51}$$

$$\int_{\text{ABQP}} \mathbf{A}.\mathrm{d}\mathbf{s} = [\mathrm{A}_x]_{\text{ABQP}}\, \Delta y\, \Delta z \tag{1.52}$$

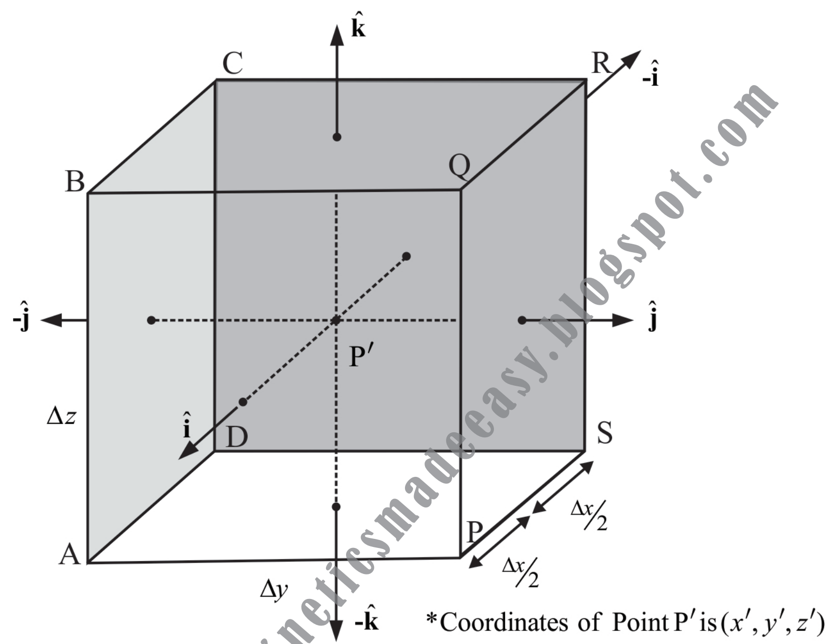


Fig.1.24

Because the box is infinitesimal the value of A_x at the center of the ABQP face can be taken to be average value of A_x over the entire face. Hence equation (1.52) is

$$\int_{\text{ABQP}} \mathbf{A}.\mathrm{d}\mathbf{s} = [\mathrm{A}_x\, \Delta y\, \Delta z]_{\text{at}\left(x'+\frac{\Delta x}{2}, y', z'\right)} \tag{1.53}$$

Applying Taylor's series expansion and neglecting higher order terms because the box is infinitesimal

$$[\mathrm{A}_x]_{\text{at}\left(x'+\frac{\Delta x}{2}, y', z'\right)} = \mathrm{A}_x(x', y', z') + \frac{\Delta x}{2}\left(\frac{\partial \mathrm{A}_x}{\partial x}\right)_{(x', y', z')}$$

Substituting the above equation in 1.53

$$\int_{ABQP} \mathbf{A}.d\mathbf{s} = \left[A_x(x', y', z') + \frac{\Delta x}{2}\left(\frac{\partial A_x}{\partial x}\right)_{(x', y', z')} \right] \Delta y \Delta z \tag{1.54}$$

Similarly for the face CRSD

$$\int_{CRSD} \mathbf{A}.d\mathbf{s} = (\mathbf{A})_{CRSD} . \Delta \mathbf{s}_{CRSD} \tag{1.55}$$

$$= \left[A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \right]_{CRSD} . \left(\Delta y \Delta z \left(-\hat{\mathbf{i}} \right) \right) \tag{1.56}$$

$$= -\left[A_x \Delta y \Delta z \right]_{at\left(x' - \frac{\Delta x}{2}, y', z'\right)} \tag{1.57}$$

Using Taylor series expansion and neglecting higher order terms

$$\left[A_x \right]_{at\left(x' + \frac{\Delta x}{2}, y', z'\right)} = A_x(x', y', z') - \frac{\Delta x}{2}\left(\frac{\partial A_x}{\partial x}\right)_{(x', y', z')}$$

Substituting the above equation in 1.57.

$$\int_{CRSD} \mathbf{A}.d\mathbf{s} = -\left[A_x(x', y', z') - \frac{\Delta x}{2}\left(\frac{\partial A_x}{\partial x}\right)_{(x', y', z')} \right] \Delta y \Delta z \tag{1.58}$$

Adding equations 1.54, 1.58 we get

$$\int_{ABQP} \mathbf{A}.d\mathbf{s} + \int_{CRSD} \mathbf{A}.d\mathbf{s} = \left(\frac{\partial A_x}{\partial x}\right)_{(x', y', z')} \Delta x \Delta y \Delta z \tag{1.59}$$

Applying the same procedure to ABCD, PQRS we get

$$\int_{ABCD} \mathbf{A}.d\mathbf{s} + \int_{PQRS} \mathbf{A}.d\mathbf{s} = \left(\frac{\partial A_y}{\partial y}\right)_{(x', y', z')} \Delta x \Delta y \Delta z \tag{1.60}$$

Similarly for the faces BCRQ, ADSP

$$\int_{\text{BCRQ}} \mathbf{A}.d\mathbf{s} + \int_{\text{ADSP}} \mathbf{A}.d\mathbf{s} = \left(\frac{\partial \mathrm{A}_z}{\partial z}\right)_{(x',y',z')} \Delta x \Delta y \Delta z \tag{1.61}$$

Substituting equations (1.59), (1.60), (1.61) in equation 1.49

$$\oint_{\mathrm{S}} \mathbf{A}.d\mathbf{s} = \left(\frac{\partial \mathrm{A}_x}{\partial x} + \frac{\partial \mathrm{A}_y}{\partial y} + \frac{\partial \mathrm{A}_z}{\partial \mathrm{z}}\right)_{(x',y',z')} \Delta x \Delta y \Delta z \tag{1.62}$$

With $\tau = \Delta x \Delta y \Delta z$, the volume of the box, the above equation becomes

$$\oint_{\mathrm{S}} \mathbf{A}.d\mathbf{s} = \left(\frac{\partial \mathrm{A}_x}{\partial x} + \frac{\partial \mathrm{A}_y}{\partial y} + \frac{\partial \mathrm{A}_z}{\partial z}\right)_{(x',y',z')} \tau \tag{1.63}$$

Substituting equation 1.63 in 1.48

$$\operatorname{div} \mathbf{A} = \frac{\partial \mathrm{A}_x}{\partial x} + \frac{\partial \mathrm{A}_y}{\partial y} + \frac{\partial \mathrm{A}_z}{\partial z} \tag{1.64}$$

The above equation applies for any point (x', y', z'). Equation 1.64 can be written as

$$\operatorname{div} \mathbf{A} = \left(\hat{\mathbf{i}}\frac{\partial \mathrm{A}_x}{\partial x} + \hat{\mathbf{j}}\frac{\partial \mathrm{A}_y}{\partial y} + \hat{\mathbf{k}}\frac{\partial \mathrm{A}_z}{\partial y}\right).\left(\hat{\mathbf{i}}\,\mathrm{A}_x + \hat{\mathbf{j}}\,\mathrm{A}_y + \hat{\mathbf{k}}\,\mathrm{A}_z\right) \tag{1.65}$$

$$\operatorname{div} \mathbf{A} = \nabla.\mathbf{A} \tag{1.66}$$

1.15 Vector differentiation: curl of a vector

In your daily life you could have come across the word "curly hair". The meaning of word curl is circulation. Curl of the vector is the measure of how much the given vector circulates about the given point in question.

For example, the vector shown in figure 1.25 is having high curl as it is circulating about a given point. A vector alone can curl, scalar cannot curl. For example if we consider a block of mass 200kg, it is 200kg as itself and the value

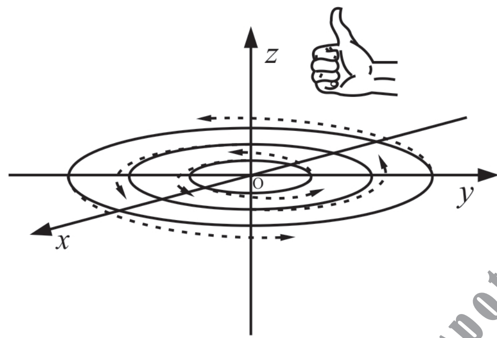


Fig.1.25

200kg doesn't circulate. On the other hand consider a swirl as shown in figure 1.26 which is having high curl or circulation. The velocity of water at each point is different and circulates about the central point. We will now see the mathematical description of curl.

The curl of the vector field at any point is a vector quantity, the magnitude of the vector quantity being given by maximum line integral per unit area, the line integral being carried out along the boundary of infinitesimal test area at that point and the direction of the vector is perpendicular to the plane of the test area.

As per the above definition the curl of any vector **A** is defined as

$$\operatorname{curl}\mathbf{A} = \lim_{\Delta s \to 0} \frac{\left[\hat{\mathbf{n}} \oint_C \mathbf{A}.d\boldsymbol{l}\right]_{max}}{\Delta s} \tag{1.67}$$

Here Δs is the test area which is enclosed by contour C as shown in figure 1.27. Here $\hat{\mathbf{n}}$ is the unit vector normal to the test area. We will see how to fix the direction of curl or $\hat{\mathbf{n}}$ later on.

Now let us define the component of curl

$$\left(\text{curl}\,\mathbf{A}\right)_C = \hat{\mathbf{n}}_C.\left(\text{curl}\,\mathbf{A}\right)$$

$$= \lim_{\Delta S_C \to 0} \frac{\oint_{C_C} \mathbf{A}.d\boldsymbol{l}}{\Delta s_C} \qquad (1.68)$$

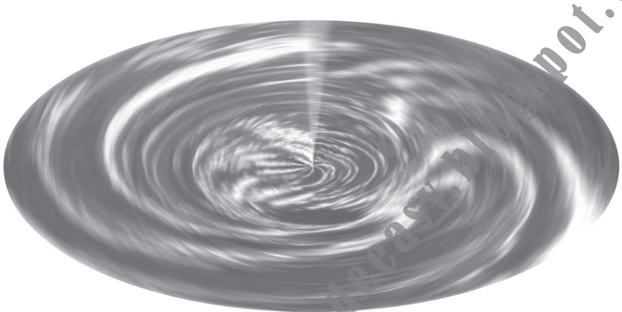


Fig.1.26

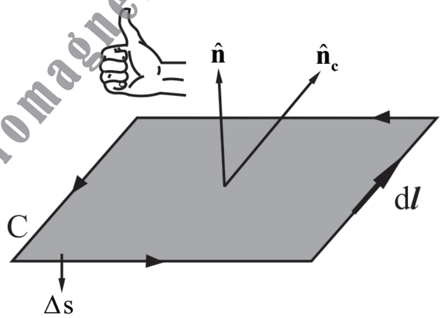


Fig.1.27

Here $\hat{\mathbf{n}}_\mathbf{c}$ is the unit vector normal to the area Δs_C, where Δs_C is the area bounded by contour C_c.

The direction of $\hat{\mathbf{n}}$ in figure 1.27 is given by right hand rule. When the fingers curl around C the thumb gives the direction of $\hat{\mathbf{n}}$ which is perpendicular to

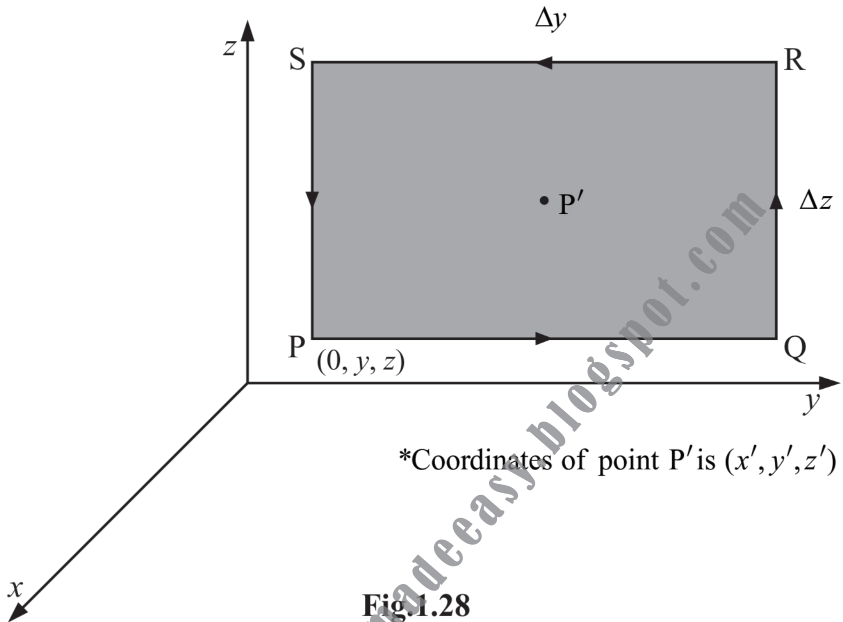


Fig.1.28

the area Δs. The same right hand rule is true for $\hat{\mathbf{n}}_{\mathbf{c}}$ which is perpendicular to area Δs_C, area Δs_C being bounded by contour C_C. Contour C_C is not shown in figure 1.27.

Consider a infinitesimal loop PQRS in the $z - y$ plane surrounding the point P' in the region of vector function **A**, as shown in figure 1.28. The positive direction around the loop is assigned by right hand rule. When the fingers of the right hand point in the direction of the arrows in the contour PQRS in figure 1.28 then the thumb gives the direction of the positive normal to the surface. This positive normal for the loop PQRS is along positive x-axis.

Now let us calculate the line integral of **A** along PQRS.

The line integral of **A** along PQ is

$$\int_{PQ} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \left[\mathrm{A}_x\,\hat{\mathbf{i}} + \mathrm{A}_y\,\hat{\mathbf{j}} + \mathrm{A}_z\,\hat{\mathbf{k}}\right].\Delta \mathrm{y}\,\hat{\mathbf{j}}$$

$$= \left[\mathrm{A}_y\,\Delta y\right]_{\text{at PQ}}$$

Because the loop is infinitesimal the value of A_x at the center of the line PQ can be taken to be average value of A_x over the entire line. Hence the above equation becomes

$$\int_{PQ} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \left[A_y\,\Delta y\right]_{\text{at}\left(x',y',z'-\frac{\Delta z}{z}\right)} \tag{1.69}$$

The line integral of **A** along RS will be

$$\int_{RS} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \left[A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}\right].\left(-\Delta y\,\hat{\mathbf{j}}\right)$$

$$= -\left[A_y\,\Delta y\right]_{\text{at}\left(x',y',z'+\frac{\Delta z}{z}\right)} \tag{1.70}$$

The line integral of **A** along SP is

$$\int_{SP} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \left[A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}\right].\left(-\Delta z\,\hat{\mathbf{k}}\right)$$

$$= -\left[A_z\,\Delta z\right]_{\text{at}\left(x',y'-\frac{\Delta y}{z},z'\right)} \tag{1.71}$$

The line integral of **A** along QR is

$$\int_{QR} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \left[A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}\right].\left(\Delta z\,\hat{\mathbf{k}}\right)$$

$$= \left[A_z\,\Delta z\right]_{\text{at}\left(x',y'+\frac{\Delta y}{z},z'\right)} \tag{1.72}$$

The line integral of $\oint_{PQRS} \mathbf{A}.\mathrm{d}\boldsymbol{l}$ can be written as

$$\oint_{PQRS} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \oint_{PQ} \mathbf{A}.\mathrm{d}\boldsymbol{l} + \oint_{RS} \mathbf{A}.\mathrm{d}\boldsymbol{l} + \oint_{SP} \mathbf{A}.\mathrm{d}\boldsymbol{l} + \oint_{QR} \mathbf{A}.\mathrm{d}\boldsymbol{l} \tag{1.73}$$

Substituting equations 1.69, 1.70, 1.71, 1.72 in equation 1.73 we get

$$\oint_{PQRS} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \left[\left(\mathrm{A}_y\right)_{\left(x',y',z'-\frac{\Delta z}{z},\right)} - \left(\mathrm{A}_y\right)_{\left(x',y',z'+\frac{\Delta z}{z},\right)} \right] \Delta y$$

$$+ \left[\left(\mathrm{A}_z\right)_{\left(x',y'+\frac{\Delta y}{z},z'\right)} - \left(\mathrm{A}_z\right)_{\left(x',y'-\frac{\Delta z}{z},z'\right)} \right] \Delta z \qquad (1.74)$$

Using Taylor's series expansion, neglecting higher order terms we get

$$\left[\left(\mathrm{A}_y\right)_{\left(x',y',z'-\frac{\Delta z}{z},\right)} - \left(\mathrm{A}_y\right)_{\left(x',y',z'+\frac{\Delta z}{z},\right)} \right] = -\frac{\partial \mathrm{A}_y}{\partial z} \Delta z \qquad (1.75)$$

and

$$\left[\left(\mathrm{A}_z\right)_{\left(x',y'+\frac{\Delta y}{z},z'\right)} - \left(\mathrm{A}_z\right)_{\left(x',y'-\frac{\Delta y}{z},z'\right)} \right] = \frac{\partial \mathrm{A}_z}{\partial_y} \Delta y \qquad (1.76)$$

Substituting 1.75 and 1.76 in 1.74

$$\oint_{PQRS} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \frac{\partial \mathrm{A}_z}{\partial y} \Delta y \Delta z - \frac{\partial \mathrm{A}_y}{\partial z} \Delta z \Delta y \qquad (1.77)$$

With $\Delta \mathrm{s} = \Delta y \Delta z$ we get

$$\oint_{PQRS} \mathbf{A}.\mathrm{d}\boldsymbol{l} = \left[\frac{\partial \mathrm{A}_z}{\partial y} - \frac{\partial \mathrm{A}_y}{\partial z} \right] \Delta \mathrm{s} \qquad (1.78)$$

Substituting 1.78 in 1.68

$$\left(\mathrm{curl}\,\mathbf{A}\right)_x = \hat{\mathbf{i}}.\left(\mathrm{curl}\,\mathbf{A}\right)$$

$$= \lim_{\Delta \mathrm{s} \to 0} \frac{\oint_{PQRS} \mathbf{A}.\mathrm{d}\boldsymbol{l}}{\Delta \mathrm{s}} = \left[\frac{\partial \mathrm{A}_z}{\partial y} - \frac{\partial \mathrm{A}_y}{\partial z} \right] \qquad (1.79)$$

Similarly

$$\left(\text{curl}\,\mathbf{A}\right)_y = \frac{\partial \text{A}_x}{\partial z} - \frac{\partial \text{A}_z}{\partial x} \tag{1.80}$$

and

$$\left(\text{curl}\,\mathbf{A}\right)_z = \frac{\partial \text{A}_y}{\partial x} - \frac{\partial \text{A}_x}{\partial y} \tag{1.81}$$

We know that

$$\left(\text{curl}\,\mathbf{A}\right) = \left(\text{curl}\,\mathbf{A}\right)_x \hat{\mathbf{i}} + \left(\text{curl}\,\mathbf{A}\right)_y \hat{\mathbf{j}} + \left(\text{curl}\,\mathbf{A}\right)_z \hat{\mathbf{k}} \tag{1.82}$$

Substituting equation 1.79, 1.80, 1.81 in equation 1.82

$$\text{curl}\,\mathbf{A} = \left[\frac{\partial \text{A}_z}{\partial y} - \frac{\partial \text{A}_y}{\partial z}\right]\hat{\mathbf{i}} + \left[\frac{\partial \text{A}_x}{\partial z} - \frac{\partial \text{A}_z}{\partial x}\right]\hat{\mathbf{j}} + \left[\frac{\partial \text{A}_y}{\partial x} - \frac{\partial \text{A}_x}{\partial y}\right]\hat{\mathbf{k}} \tag{1.83}$$

The above equation can be written as

$$\text{curl}\,\mathbf{A} = \left[\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right] \times \left[\text{A}_x \hat{\mathbf{i}} + \text{A}_y \hat{\mathbf{j}} + \text{A}_z \hat{\mathbf{k}}\right] \tag{1.84}$$

$$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} \tag{1.85}$$

In determinant from $\nabla \times \mathbf{A}$ is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{A}_x & \text{A}_y & \text{A}_z \end{vmatrix} \tag{1.86}$$

We have seen three different types of vector differentiation.

1. Gradient: The operator ∇ operates on an scalar function. In Cartesian coordinates

$$\nabla \text{u} = \hat{\mathbf{i}}\frac{\partial \text{u}}{\partial x} + \hat{\mathbf{j}}\frac{\partial \text{u}}{\partial y} + \hat{\mathbf{k}}\frac{\partial \text{u}}{\partial z}$$

As seen above gradient of a scalar function is a vector.

Divergence: The operator ∇ operates on a vector by means of dot product. In Cartesian coordinates

$$\nabla . \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

As seen above divergence of a vector function is a scalar.

Curl: The operator ∇ operates on a vector by means of cross product. In Cartesian coordinates

$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{\mathbf{i}} + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{\mathbf{j}} + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{\mathbf{k}}$$

As seen above the curl of a vector function is a vector. Because curl of any vector is a vector it must have both magnitude and direction. Let us see an example.

In figure 1.25 let the velocity vector **v** given by $v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\,\hat{\mathbf{k}}$

be curling around with O as center. Let us define curl of **v** as

$$\mathbf{C} = \nabla \times \mathbf{v} = \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right]\hat{\mathbf{i}} + \left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right]\hat{\mathbf{j}} + \left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]\hat{\mathbf{k}}$$

Then the direction of **C**, the curl of **v** is given by right hand rule. When the fingers of right hand point in the direction of circulating **v** as shown in figure 1.25 then the thumb gives the direction of **C** which is along *z*-axis. Thus there are two vectors **v**, **C**. **v** the velocity which is circulating about and **C**, the curl of **v** pointing along *z* direction.

2. We have seen that divergence of any vector means how much the vector is spreading out. Curl of any vector means how much the vector is circulating about.

   If $\nabla . \mathbf{A} = 0$ then **A** is called solenoidal vector

   If $\nabla \times \mathbf{A} = 0$ then **A** is known as irrotational vector.

For all the vectors shown in figure 1.22 curl of these vectors is zero because nothing is circulating. For the vector shown in figure 1.25 divergence is zero because nothing is spreading out.

3. Divergence and curl operate on a vector. As we have already seen divergence and curl of a scalar doesn't have a meaning.

Example 1.5

Find the gradient of u at the point (1, 2, 3) if u = $x^2 - yz$

Solution:

$$\nabla \mathrm{u} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial x}\right)\left(x^2 - yz\right)$$

$$\Rightarrow \nabla \mathrm{u} = 2x\hat{\mathbf{i}} + z\hat{\mathbf{j}} + y\hat{\mathbf{k}}$$

Gradient of a scalar function u, $\nabla \mathrm{u}$ is a vector as we discussed previously.

$\nabla \mathrm{u}$ at the point (1, 2, 3) is then

$$\nabla \mathrm{u} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Example 1.6

Find the divergence of the vector $\mathbf{A} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \mathbf{B} = x^2 y\hat{\mathbf{i}} + z^3 y\hat{\mathbf{j}}$

Solution:

$$\nabla.\mathbf{A} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial y}\right).\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)$$

$$\nabla.\mathbf{A} = 3$$

$$\nabla.\mathbf{B} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial y}\right).\left(x^2 y\hat{\mathbf{i}} + z^3 y\hat{\mathbf{k}}\right)$$

$$\nabla.\mathbf{B} = 2xy + z^3$$

Divergence of **A** is 2, divergence of **B** is $2xy + z^3$.

Example 1.7

Calculate curl of vector $\mathbf{A} = y\hat{\mathbf{k}}$

Solution:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & y \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{i}}\left[\frac{\partial y}{\partial y} - 0\right] - \hat{\mathbf{j}}[0-0] + \hat{\mathbf{k}}[0-0]$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{i}}$$

Example 1.8

Prove that for any vector **A**, div curl **A** = 0. That is prove that $\nabla.(\nabla \times \mathbf{A}) = 0$

$$\nabla.(\nabla \times \mathbf{A}) = \nabla. \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{A}_x & \mathrm{A}_y & \mathrm{A}_z \end{vmatrix}$$

$$= \left[\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right].\left[\hat{\mathbf{i}}\left(\frac{\partial \mathrm{A}_z}{\partial y} - \frac{\partial \mathrm{A}_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial \mathrm{A}_x}{\partial z} - \frac{\partial \mathrm{A}_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial \mathrm{A}_y}{\partial x} - \frac{\partial \mathrm{A}_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{A}_z}{\partial y} - \frac{\partial \mathrm{A}_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathrm{A}_x}{\partial z} - \frac{\partial \mathrm{A}_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathrm{A}_y}{\partial x} - \frac{\partial \mathrm{A}_x}{\partial y}\right)$$

$$= \frac{\partial^2 \mathrm{A}_z}{\partial x\,\partial y} - \frac{\partial^2 \mathrm{A}_y}{\partial x\,\partial z} + \frac{\partial^2 \mathrm{A}_x}{\partial y\,\partial z} - \frac{\partial^2 \mathrm{A}_z}{\partial y\,\partial x} + \frac{\partial^2 \mathrm{A}_y}{\partial z\,\partial x} - \frac{\partial^2 \mathrm{A}_x}{\partial z\,\partial y} = 0$$

Example 1.9

Prove that curl grad u = 0. That is prove that $\nabla \times \nabla \mathrm{u} = 0$ where u is a scalar function.

Solution:

$$\nabla \times \nabla \mathrm{u} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \mathrm{u}}{\partial x} & \frac{\partial \mathrm{u}}{\partial y} & \frac{\partial \mathrm{u}}{\partial z} \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(\frac{\partial^2 \mathrm{u}}{\partial y\,\partial z} - \frac{\partial^2 \mathrm{u}}{\partial z\,\partial y}\right) + \hat{\mathbf{j}}\left(\frac{\partial^2 \mathrm{u}}{\partial z\,\partial x} - \frac{\partial^2 \mathrm{u}}{\partial x\,\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial^2 \mathrm{u}}{\partial x\,\partial y} - \frac{\partial^2 \mathrm{u}}{\partial y\,\partial x}\right) = 0$$

1.16 Divergence Theorem

Divergence theorem is helpful in transforming a surface integral of a vector [flux of the vector] into a volume integral of the divergence of the vector and vice versa. The theorem states that the volume integral of the divergence of the vector **A** is equal to the flux of the vector over the closed surface S enclosing the volume τ over which the volume integral is calculated i-e,

$$\int_{\tau} \nabla.\mathbf{A}\,d\tau = \oint_{S} \mathbf{A}.\mathbf{ds} \qquad (1.87)$$

Divergence theorem is also known as Gauss divergence theorem.

We have already seen that divergence of a vector is a scalar quantity [see section (1.15)].

The volume τ is also a scalar quantity. Hence when dealing with integral $\int_{\tau} \nabla.\mathbf{A}\,d\tau$ we will be working with scalars on the other hand in the integral

$\oint_{S} \mathbf{A}.\mathbf{ds}$, surface d**s** is a vector quantity. Hence when working integral $\oint_{S} \mathbf{A}.\mathbf{ds}$ we

will be working with vectors. This point will be more clear in example (1.10).

Now we shall see the proof for divergence theorem.

Let us consider some arbitrary volume τ which is enclosed by surface S as shown in figure (1.29a). The arbitrary volume is present in a region of vector function **A**. The flux diverging from the surface S of volume τ is

$$\phi = \oint_S \mathbf{A}.d\mathbf{s} \tag{1.88}$$

Now let us divide the volume τ into two parts with volume τ_1 and τ_2 which are enclosed by surfaces S_1 and S_2 respectively as shown in figure (1.29b).

We see that in figure 1.29b at the common internal surface for S_1 and S_2 the outward drawn normal's of the two parts point in opposite direction. Hence the contribution of the internal common surface to the flux of the two parts will cancel each other.

The flux emerging out of surface $S_1 = \oint_{S_1} \mathbf{A}.d\mathbf{s}_1$ (1.89)

The flux emerging out of surface $S_2 = \oint_{S_2} \mathbf{A}.d\mathbf{s}_2$ (1.90)

Because the flux due to the common internal surface for S_1 and S_2 cancel out each other, the flux from the rest of the surfaces of S_1 and S_2 will be equal to the flux from the original surface S. Thus

$$\phi = \oint_S \mathbf{A}.d\mathbf{s} = \oint_{S_1} \mathbf{A}.d\mathbf{s}_1 + \oint_{S_2} \mathbf{A}.d\mathbf{s}_2 \tag{1.91}$$

Suppose we divide the volume τ into a large number of volumes $\tau_1, \tau_2 .... \tau_n$ which are enclosed by the surfaces S_1, S_2, S_3, ...S_n respectively then

$$\phi = \oint_S \mathbf{A}.d\mathbf{s} = \oint_{S_1} \mathbf{A}.d\mathbf{s}_1 + \oint_{S_2} \mathbf{A}.d\mathbf{s}_2 + ... + \oint_{S_n} \mathbf{A}.d\mathbf{s}_n \tag{1.92}$$

$$\phi = \oint_S \mathbf{A}.d\mathbf{s} = \sum_{j=1}^{N} \oint_{S_j} \mathbf{A}.d\mathbf{s}_j \tag{1.93}$$

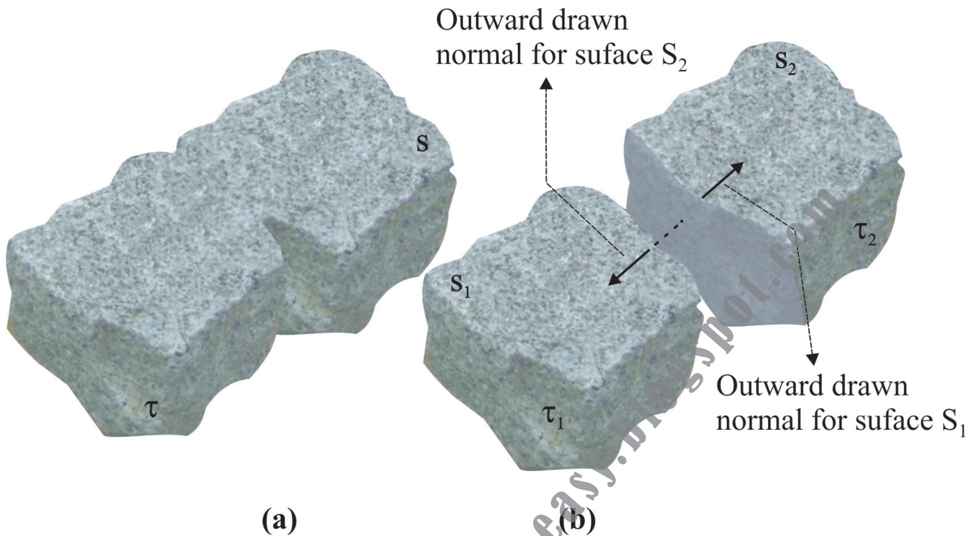


Fig.1.29

The above equation can be written as

$$\oint_S \mathbf{A}.d\mathbf{s} = \sum_{j=1}^{N} \tau_j \frac{\oint_{S_j} \mathbf{A}.d\mathbf{s}_j}{\tau_j} \tag{1.94}$$

The volume τ_j is infinitely small if N is very large. i.e., if $N \to \infty, \tau_j \to 0$.

In this limit using equation 1.48 equation 1.94 can be written as

$$\oint_S \mathbf{A}.d\mathbf{s} = \sum_{j=1}^{N} \Delta\tau_j \; \nabla.\mathbf{A} \tag{1.95}$$

We have $\Delta\tau_j$ instead of τ_j in the above equation because volume τ_j is infinitely small.

Converting the summation symbol into integration and expressing $\Delta\tau_j$ as $d\tau$ we can write the above equation as

$$\oint_S \mathbf{A}.d\mathbf{s} = \int_\tau \nabla.\mathbf{A}\, d\tau \tag{1.96}$$

Hence divergence theorem is proved. Divergence theorem is an important identity in vector analysis. We will use the theorem to establish other theorems in electromagnetics.

Now we will see the physical interpretation of divergence theorem. In figure 1.23a we have shown a pipe fitted in a floor and water spreads out from the floor. We used this example to explain that whenever there is divergence it implicitly means that there is a source present. Let us restate the above example in a slightly different manner. Instead of fixing the pipe in the floor, let us fix two pipes inside a cubical structure which will allow water to pass through.[See figure 1.30].

Assuming that water is incompressible whatever water comes out of the pipes flows out through the six faces of the cube.

Now let us generalize the above example to any given vector. Consider figure 1.31(Figure 1.31 can be of any irregular shape). There are n numbers of sources giving rise to divergence. The n numbers of sources are contained in volume τ which is bounded by surface S. In section 1.12 we saw that $\oint \mathbf{A}.\mathbf{ds}$ is the flux of the vector. Just as in the figure 1.30 whatever water comes out of the pipe flows

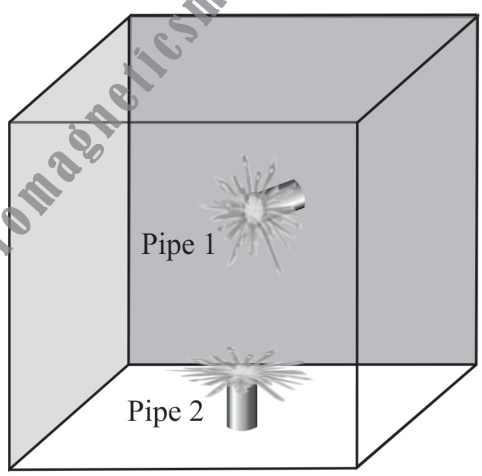


Fig.1.30

out of the surfaces of the cube, similarly in figure 1.31 whatever diverges from the n number of sources $\left(= \int_{\tau} \nabla.\mathbf{A}\,d\tau\right)$ flows out of surface in the form of flux $\left[= \oint \mathbf{A}.\mathbf{ds}\right]$.

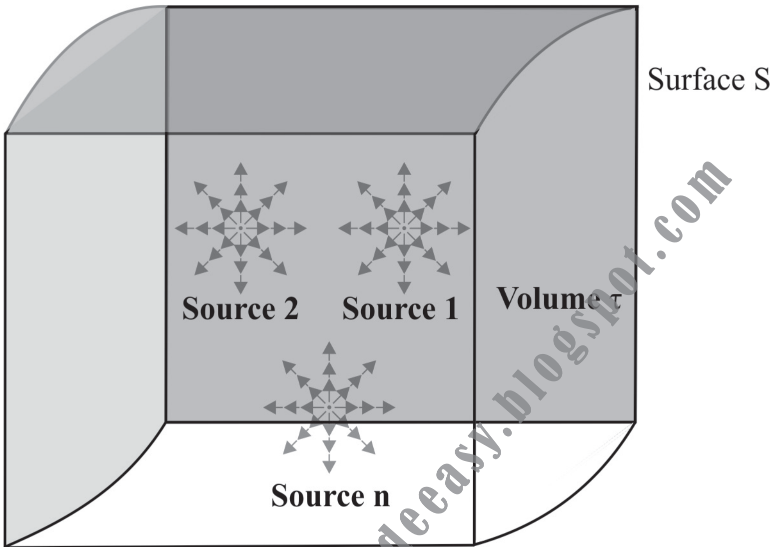


Fig.1.31

Hence $\int_{\tau} \nabla.\mathbf{A}\,d\tau = \oint_{S} \mathbf{A}.d\mathbf{s}$

Example 1.10

Verify divergence theorem for the vector

$\mathbf{A} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

for a unit cube which is located at the origin.

Solution:

$$\int_{\tau} \nabla.\mathbf{A}\,d\tau = \oint_{S} \mathbf{A}.d\mathbf{s}$$

First let us evaluate the integral $\int_{\tau} \nabla.\mathbf{A}\,d\tau$

$$\nabla.\mathbf{A} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)$$

$$=1 + 1 + 1 + = 3$$

Now

$$\int_{\tau} \nabla.\mathbf{A}\,\mathrm{d}\tau = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= 3[x]_0^1\,[y]_0^1\,[z]_0^1 = 3$$

Now let us evaluate the surface integral. The cube has six faces. So the surface integral has to be evaluated at all the six faces, shown in figure 1.32

i) For face ABCD, $x = 0$ ds $= -\mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{i}}$ as outward drawn normal to ABCD is along $-x$ axis

$$\iint_{\substack{\text{ABCD}\\ x=0}} \mathbf{A}.\mathrm{d}y\,\mathrm{d}z\,(-\hat{\mathbf{i}})$$

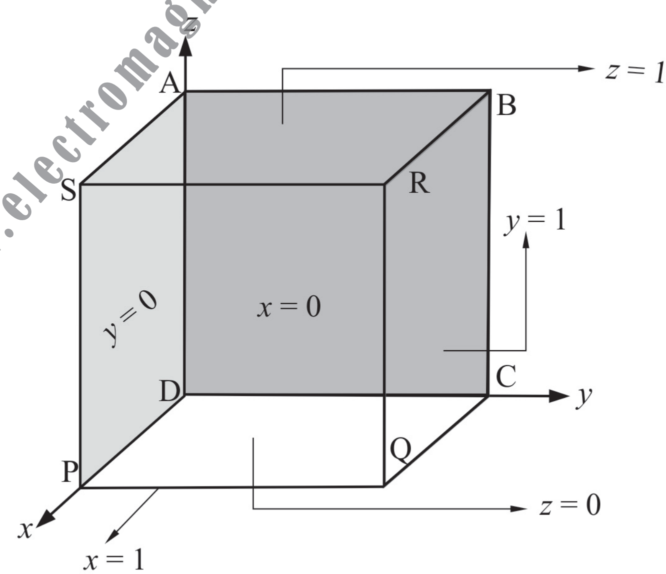


Fig.1.32

$$= \iint\limits_{\substack{\text{ABCD}\\ x=0}} \left(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}\right).\mathrm{d}y\,\mathrm{d}z\,(-\hat{\mathbf{i}})$$

$$= \iint\limits_{\substack{\text{ABCD}\\ x=0}} \left(-x\,\mathrm{d}y\,\mathrm{d}z\right)=0 \text{ as } x=0$$

ii) For the face PQRS, x = 1 and ds = dy dz $\hat{\mathbf{i}}$. Hence

$$\iint\limits_{\substack{\text{PQRS}\\ x=1}} \mathbf{A}.\mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{i}} = \iint\limits_{\substack{\text{PQRS}\\ x=1}} \left(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}\right).\mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{i}}$$

$$\iint\limits_{\substack{\text{PQRS}\\ x=1}} x\,\mathrm{d}y\,\mathrm{d}z=1$$

iii) For the face SADP, y = 0 and ds = –dx dz $\hat{\mathbf{j}}$ as the outward drown normal to SADP is along –y axis

$$= \iint\limits_{\substack{\text{SADP}\\ y=0}} \mathbf{A}.\mathrm{d}x\,\mathrm{d}z\,(-\hat{\mathbf{j}})$$

$$= \iint\limits_{\substack{\text{SADP}\\ y=0}} \left(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}\right).\,(-\mathrm{d}x\,\mathrm{d}z)\,\hat{\mathbf{j}}$$

$$= \iint\limits_{\substack{\text{SADP}\\ y=0}} y\,(-\mathrm{d}x\,\mathrm{d}z)=0 \text{ as } y=0$$

iv) For the face RBCQ, y = 1 and ds = dx dz $\hat{\mathbf{j}}$

$$= \iint\limits_{\substack{\text{RBCQ}\\ y=1}} \mathbf{A}.\left(\mathrm{d}x\,\mathrm{d}z\right)\hat{\mathbf{j}}$$

$$= \iint_{\substack{\text{RBCQ} \\ y=1}} \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right).\,(\mathrm{d}x\,\mathrm{d}z)\,\hat{\mathbf{j}}$$

$$= \iint_{\substack{\text{RBCQ} \\ y=1}} y\,\mathrm{d}x\,\mathrm{d}z = 1$$

v) For the face PQCD, $z = 0$ and ds = dx dy $(-\hat{\mathbf{k}})$ as the outward drawn normal is along $-z$ axis

$$= \iint_{\substack{\text{PQCD} \\ z=0}} \mathbf{A}.\left(\mathrm{d}x\,\mathrm{d}y\right)(-\hat{\mathbf{k}})$$

$$= \iint_{\substack{\text{PQCD} \\ z=0}} \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right).(-\mathrm{d}x\,\mathrm{d}y)\hat{\mathbf{k}}$$

$$= \iint_{\substack{\text{PQCD} \\ z=0}} (-z\ \mathrm{d}x\,\mathrm{d}y) = 0 \text{ as } z = 0.$$

vi) For the face SRBA, $z = 1$ and ds = dx dy $\hat{\mathbf{k}}$

$$= \iint_{\substack{\text{SRBA} \\ z=1}} \mathbf{A}.\,\mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{k}}$$

$$= \iint_{\substack{\text{SRBA} \\ z=1}} \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right).\mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{k}}$$

$$= \iint_{\substack{\text{SRBA} \\ z=1}} z\ \mathrm{d}x\,\mathrm{d}y = 1$$

$$\text{Thus} \oint \mathbf{A}.\mathbf{ds} = \int_{\text{ABCD}} \mathbf{A}.\mathbf{ds} + \int_{\text{PQRS}} \mathbf{A}.\mathbf{ds}$$

$$+ \int_{\text{SADP}} \mathbf{A}.\mathbf{ds} + \int_{\text{RBCQ}} \mathbf{A}.\mathbf{ds} + \int_{\text{PQCD}} \mathbf{A}.\mathbf{ds} + \int_{\text{SRBA}} \mathbf{A}.\mathbf{ds} = 3$$

Hence divergence theorem is verified. We evaluated two integrals $\int_\tau \nabla.\mathbf{A}\,d\tau$ and $\oint \mathbf{A}.\mathbf{ds}$. To evaluate $\int_\tau \nabla.\mathbf{A}\,d\tau$ we did single integration. However to evaluate $\oint \mathbf{A}.\mathbf{ds}$ we performed six integrations. $\nabla.\mathbf{A}$ and volume τ are scalar quantities. In $\int_\tau \nabla.\mathbf{A}\,d\tau$ thus only scalar quantities are involved which doesn't have direction. Hence single integration is sufficient to arrive at result. However in the integral $\oint \mathbf{A}.\mathbf{ds}$ surface is a vector quantity which involves direction. For the cube shown in figure 1.31 there are six different faces or surfaces each having its own direction. Hence six different integration for six different directions. Thus evaluation of $\oint \mathbf{A}.\mathbf{ds}$ becomes lengthy and laborious. $\int_\tau \nabla.\mathbf{A}\,d\tau$ is easy as it involves single integration because the integration involves scalars. $\oint \mathbf{A}.\mathbf{ds}$ is lengthy as it involves vectors. The above example apart from verifying divergence theorem proves the advantage of working with scalars as compared to working with vectors. Scalars are easy to work with as compared to vectors, as scalars don't involve direction.

1.17 Stoke's Theorem

Stoke's theorem is helpful in transforming a surface integral of a vector into a line integral around the boundary C, where C is the boundary of the respective surface. The theorem states that

$$\int_S \text{curl}\,\mathbf{A}.d\mathbf{s} = \oint_C \mathbf{A}.d\boldsymbol{l} \qquad (1.97)$$

Here C is the boundary of the surface S. $d\mathbf{s}, d\boldsymbol{l}$ are the small elements of S and C respectively.

Both line and surface integrals involve vectors. But however as will see in example (1.11) line integrals are easy to evaluate as compared to surface integrals.

Now let us see the proof for Stoke's theorem

Consider $\oint_C \mathbf{A}.d\boldsymbol{l}$ where C is the boundary of surface S as shown in figure 1.33a in the region of vector **A**. Let us divide the surface S in two parts S_1, S_2 having boundaries C_1 and C_2 respectively as shown in figure 1.33b. The line integral of the vector **A** over the boundaries C_1 and C_2 can be written as $\oint_{C_1} \mathbf{A}.d\boldsymbol{l}_1$ and $\oint_{C_2} \mathbf{A}.d\boldsymbol{l}_2$.

From figure 1.33b as we traverse the line PQ along boundary C_1 and line PQ along boundary C_2, the line integral of A is equal and opposite and cancel out each other, the remaining boundaries being same as that of C. Hence

$$\oint_C \mathbf{A}.d\boldsymbol{l} = \oint_{C_1} \mathbf{A}.d\boldsymbol{l}_1 + \oint_{C_2} \mathbf{A}.d\boldsymbol{l}_2 \qquad (1.98)$$

In figure 1.33c we divide the surface S into number of surface s_1, s_2, s_3 .... s_N whose boundaries are C_1, C_2, C_3 .... C_N. A small part of figure 1.33c is shown in figure 1.34. Clearly at the common interface the line integral of **A** cancels out. Using the same argument as we did for figure 1.33b to arrive at equation 1.98 we can write for figure 1.33c

$$\oint_C \mathbf{A}.d\boldsymbol{l} = \oint_{C_1} \mathbf{A}.d\boldsymbol{l}_1 + \oint_{C_2} \mathbf{A}.d\boldsymbol{l}_2 + ... + \oint_{C_N} \mathbf{A}.d\boldsymbol{l}_N \qquad (1.99)$$

$$\Rightarrow \oint_C \mathbf{A}.d\boldsymbol{l} = \sum_{j=1}^{N} \oint_{Cj} \mathbf{A}.d\boldsymbol{l}_j \qquad (1.100)$$

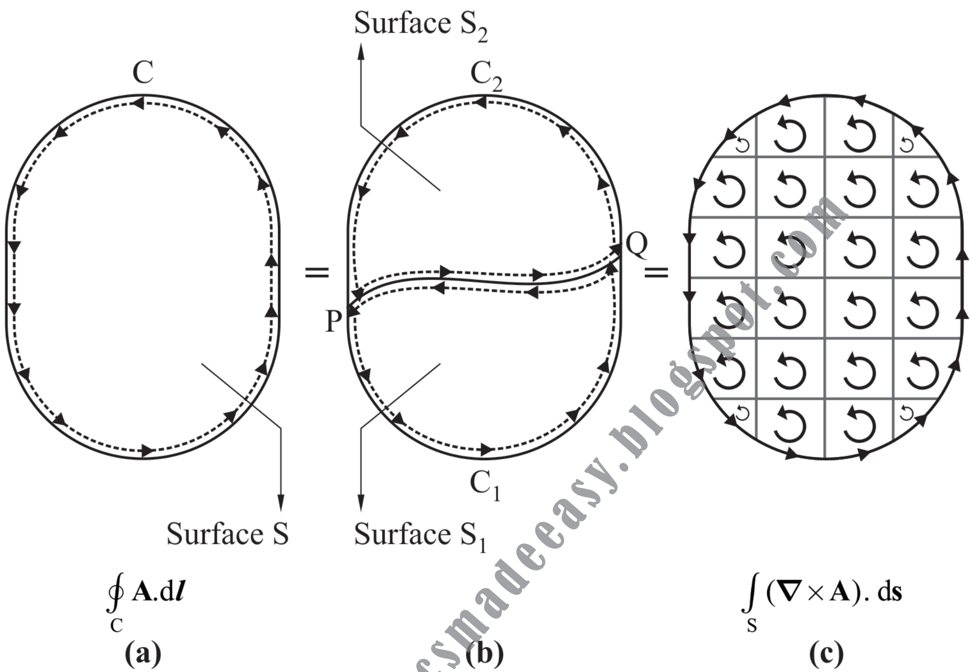


Fig.1.33

The above equation can be written as

$$\oint_C \mathbf{A}.d\boldsymbol{l} = \sum_{j=1}^{N} s_j \frac{\oint_{Cj} \mathbf{A}.d\boldsymbol{l}_j}{s_j} \tag{1.101}$$

Suppose if N is very large then the surface s_j becomes very small. i.e., If $N \to \infty, s_j \to 0$ so that we can write equation 1.68

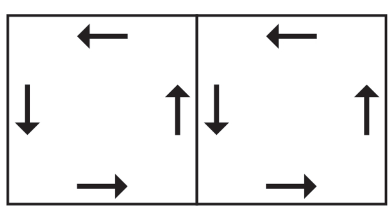


Fig.1.34

$$\lim_{\Delta s_j \to 0} \frac{\oint_{Cj} \mathbf{A}.d\boldsymbol{l}_j}{\Delta s_j} = \text{curl}\,\mathbf{A}.\hat{\mathbf{n}}_j \tag{1.102}$$

Where $\hat{\mathbf{n}}_j$ is the unit vector normal to surface Δs_j.

In the limit $N \to \infty, s_j \to 0$ equation 1.101 can be written as

$$\oint_C \mathbf{A}.d\boldsymbol{l} = \sum_{j=1}^{N} \Delta s_j \frac{\oint_{Cj} \mathbf{A}.d\boldsymbol{l}_j}{\Delta s_j} \tag{1.103}$$

Here we have used Δs_j instead of s_j to note that $s_j \to 0$ in the limit .

Using equation 1.102 in equation 1.103

$$\oint_C \mathbf{A}.d\boldsymbol{l} = \sum_{j=1}^{N} \Delta s_j \;\text{curl}\,\mathbf{A}.\hat{\mathbf{n}}_j \tag{1.104}$$

Replacing the summation by integration and Δs_j by ds, $\hat{\mathbf{n}}_j$ by $\hat{\mathbf{n}}$

$$\oint_C \mathbf{A}.d\boldsymbol{l} = \int_S \text{curl}\,\mathbf{A}.\hat{\mathbf{n}}\,ds \tag{1.105}$$

$$\Rightarrow \oint_C \mathbf{A}.d\boldsymbol{l} = \int_S \text{curl}\,\mathbf{A}.\mathbf{ds} \tag{1.106}$$

Hence Stoke's theorem is proved.

The physical interpretation of Stoke's theorem is as follows. We know that $\nabla\times\mathbf{A}$ is the measure of how much the vector **A** circulates about. As shown in figure 1.33c there is large circulation over surface S. Hence $\int_S \nabla\times\mathbf{A}.\mathbf{ds}$ is the measure of total circulation over the surface S. However as shown in figure 1.34 at the common interface the line integral of **A** cancel's out. So in figure 1.33c at all common interfaces the circulations are in opposite direction and cancel out. Hence

finally we arrive at figure 1.33a. In figure 1.33a we calculate the line integral $\oint_C \mathbf{A}.\mathrm{d}\boldsymbol{l}$ which is equivalent to $\int_S (\nabla\times\mathbf{A}).\mathbf{ds}$ as the above argument shows.

Thus to calculate the curl over the surface S in figure 1.33c you can go over the entire surface S and calculate total curl using the relation $\int_S (\nabla\times\mathbf{A}).\mathbf{ds}$ (or) go over the entire boundary C and calculate the circulation along the boundary C in figure 1.33a using the relation $\oint_C \mathbf{A}.\mathrm{d}\boldsymbol{l}$.

However whlie integrating the line integral which way to go around either clockwise or anticlockwise, along the boundary C of the figure 1.33a. The direction over which the boundary C is to be traversed is fixed by the direction of d**s** in figure 1.33. Suppose assume that d**s** in figure 1.33 is outward perpendicular to the plane of the paper. Let the thumb of your right hand point along d**s**. i.e, along the outward perpendicular to the plane of the paper. Then the fingers of right hand will be along the arrow marks along the boundary shown in figure 1.33a which is the direction in which the integration of line integral $\oint_C \mathbf{A}.\mathrm{d}\boldsymbol{l}$ is to be carried out.

Example 1.11

Verify Stoke's theorem for the surface shown in figure 1.35 if $\mathbf{A}=4\hat{\mathbf{i}}+z^2y^3\hat{\mathbf{j}}$. The square surface is of unit dimension.

Solution:

First let us calculate the line integral

(a) For the line a shown in figure 1.35 $x = 0$, $y=0\rightarrow 1$, $z = 0$

Hence $\mathbf{A} = 4\hat{\mathbf{i}}$

$$\mathbf{A}.\mathrm{d}\boldsymbol{l} = 4\mathrm{d}x \Rightarrow \int \mathbf{A}.\mathrm{d}\boldsymbol{l} = \int 4\mathrm{d}x = 0 \text{ as } \mathrm{d}x = 0$$

(b) For the line b shown in figure 1.35 $x = 0, y = 1,\ z = 0\rightarrow 1$ Hence $\mathbf{A} = 4\hat{\mathbf{i}}$

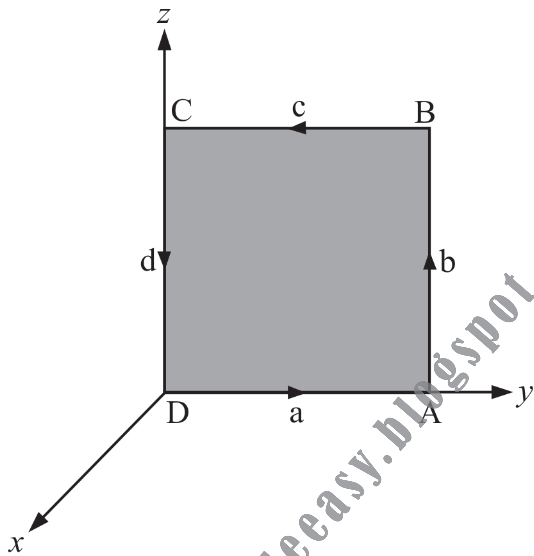
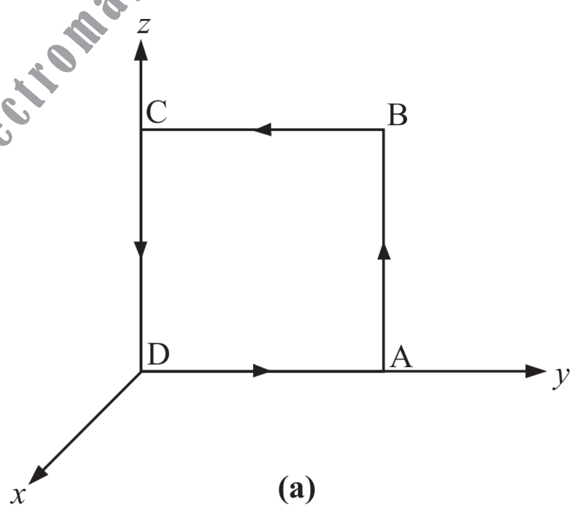


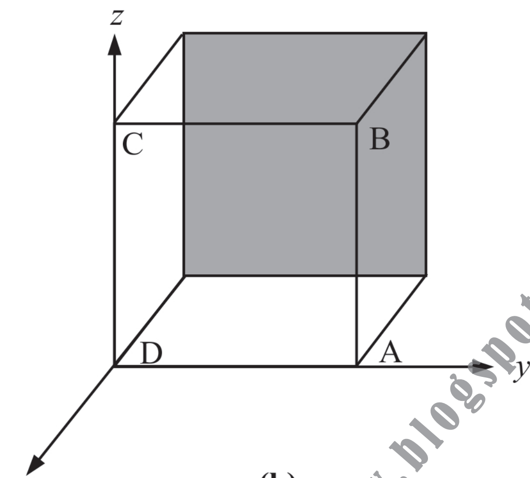
Fig.1.35

$$\int \mathbf{A}.\mathrm{d}\boldsymbol{l} = \int 4\,\mathrm{d}x = 0 \text{ as } \mathrm{d}x = 0$$

(c) For the line c shown in figure 1.35 $x = 0$, $y = 1 \rightarrow 0$, $z = 1$
Hence $\mathbf{A} = 4\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}}$



(a)
Fig.1.36



(b)

Fig.1.36

$$\mathbf{A}.\mathrm{d}\boldsymbol{l} = 4\,\mathrm{d}x + y^3\,\mathrm{d}y = y^3\,\mathrm{d}y \text{ as } \mathrm{d}x = 0$$

$$\mathbf{A}.\mathrm{d}\boldsymbol{l} = \int_1^0 y^3\,\mathrm{d}y = \left.\frac{y^4}{4}\right|_1^0 = \frac{-1}{4}$$

(d) For the line d shown in figure 1.35 $x = 0$, $y = 0$, $z = 1 \rightarrow 0$

$\mathbf{A} = 4\hat{\mathbf{i}}$

$$\int \mathbf{A}.\mathrm{d}\boldsymbol{l} = 0 \text{ as } \mathrm{d}x = 0$$

Hence the total sum $\oint \mathbf{A}.\mathrm{d}\boldsymbol{l} = 0 + 0 - \frac{1}{4} + 0 = -\frac{1}{4}$

We have $\nabla \times \mathbf{A} = -22\,y^2\,\hat{\mathbf{i}}$

By right hand rule d**s** = dy dz $\hat{\mathbf{i}}$

Thus $\int_S (\nabla \times \mathbf{A}).\mathrm{d}\mathbf{s} = \int_0^1 \int_0^1 \left(-2\,z\,y^3\right)\mathrm{d}y\,\mathrm{d}x = -\frac{1}{4}$

Hence Stoke's theorem is verified.

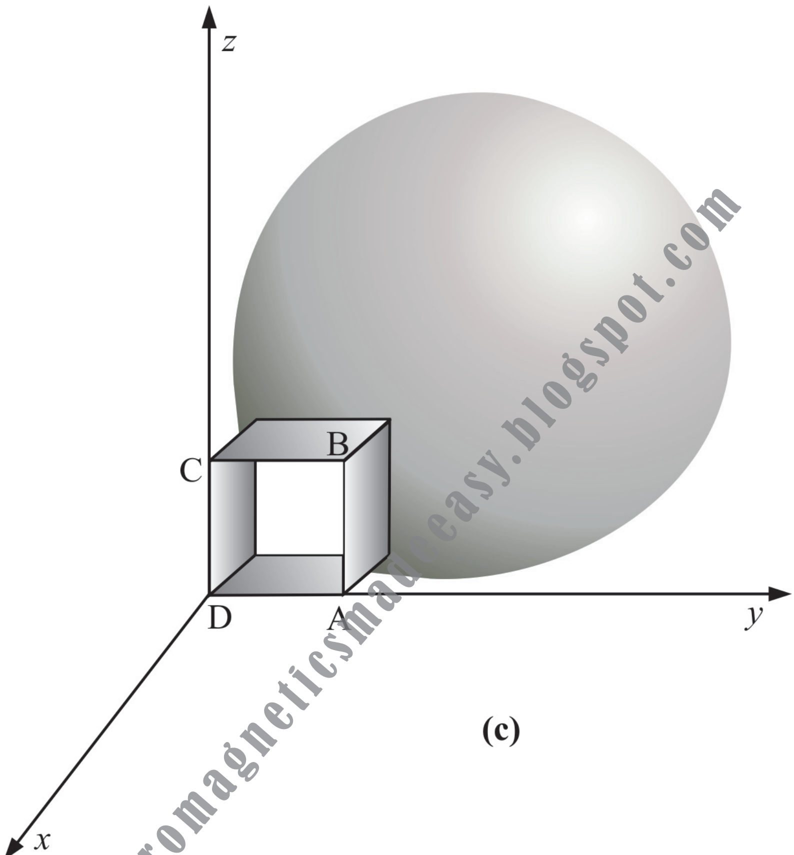


Fig.1.36

In figure 1.35 there is only one surface ABCD to work with. Suppose assume that ABCD in figure 1.35 is an elastic membrane. We have redrawn ABCD elastic membrane in figure 1.36a. Let us stretch the elastic membrane in the form of a cube as shown in figure 1.36b. As ABCD the elastic membrane is stretched into a cube, face ABCD is now open. Thus there are five surfaces of cube to work with. Hence for the given loop ABCD in figure 1.36a there is only one surface, while in figure 1.36b there are five different surfaces. For the same loop ABCD if the elastic membrane in figure 1.36a is blown up in the form of balloon as shown in figure 1.36c surface integration becomes even more complicated. In all the three figures the loop ABCD is two dimensional, just one need *z, y* axis to describe loop ABCD. But the corresponding surface associated with ABCD can be three dimensional as in figure 1.36 b, c.

1.18 The Gradient Theorem

The Gradient theorem states that

$$\int_a^b \nabla . d\boldsymbol{l} = u(b) - u(a) \tag{1.107}$$

Where u is a scalar function and a, b are end points as shown in figure 1.37

Because the integration depends on the end points a and b alone and not on the path, the integration will yield identical results for any path between a and b. For example the integration will yield same results for path 1 and path 2 in figure 1.37

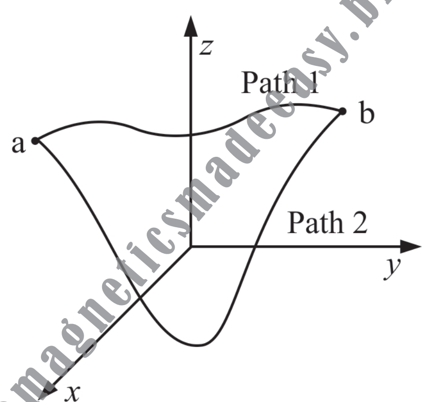


Fig.1.37

For a closed path $a = b$ and hence

$$\oint (\nabla u).d\boldsymbol{l} = u(a) - u(a) = 0 \tag{1.108}$$

1.19 Others Coordinate Systems

In two dimensions the most familiar coordinate system is cartesian coordinate system with *x*, *y* axis as shown in figure 1.38. Another coordinate system in two dimensions is polar coordinate system (r, θ). The cartesian coordinates and polar coordinates are related to each other by

$$x = r\cos\theta, y = r\sin\theta$$

The point P is figure 1.38 can be represented by either cartesian coordinates x, y or polar coordinates (r, θ). If we are interested in calculating electric (or) magnetic fields at point P due to some charge or current distributions then these physical quantities can be expressed in terms of cartesian coordinates (x, y) or polar coordinate (r, θ).

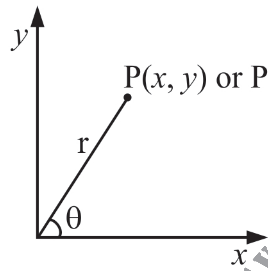


Fig.1.38

Similarly in three dimensions in addition to cartesian coordinates (x, y, z) there are number of other coordinate systems, out of which two are very important spherical coordinate system and cylindrical coordinate system.

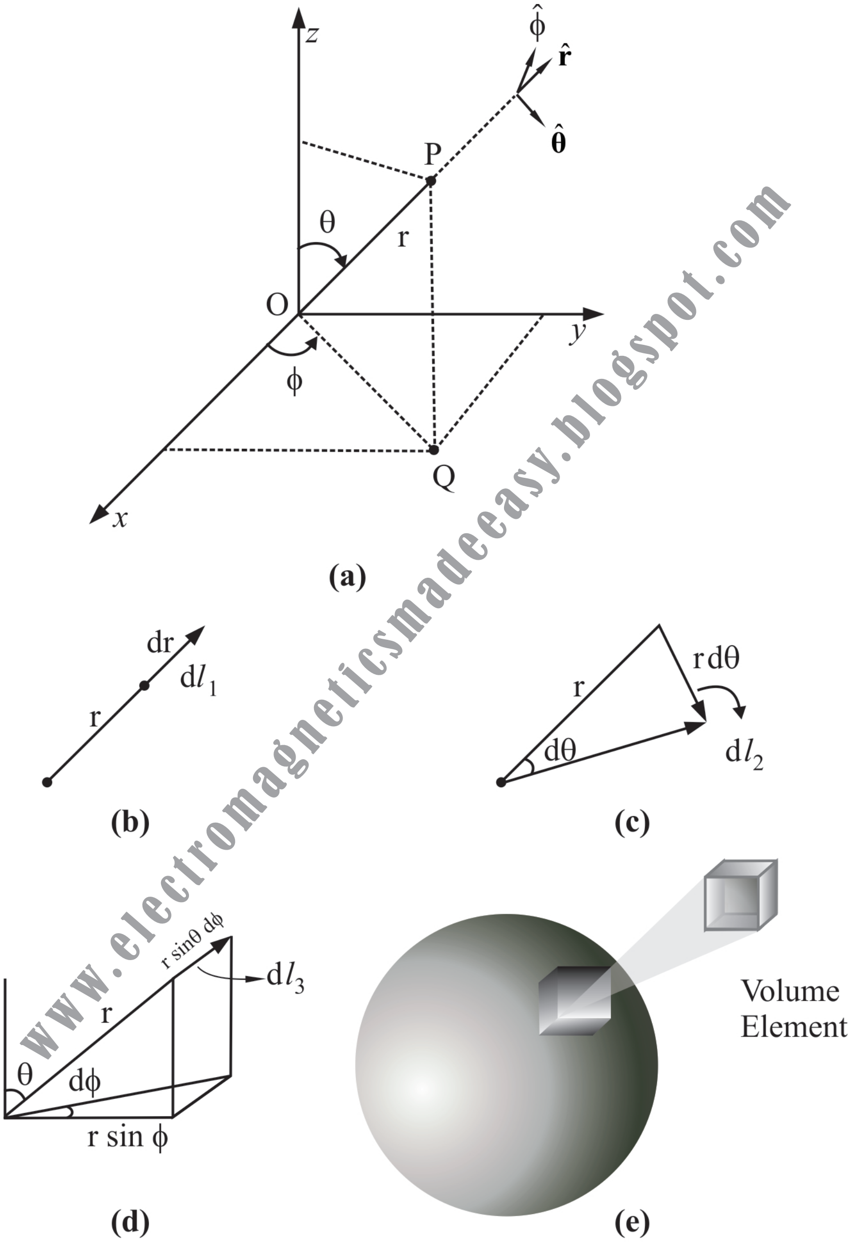
1.19a Spherical Coordinate System

In figure 1.39a we show the Cartesian coordinate system (x, y, z) and the spherical coordinate system (r, θ, ϕ). Here r is the distance from origin. θ is the angle between the z axis and the line drawn from the origin to point P. ϕ is the angle between the x axis and line O Q.

r varies from 0 to ∞, θ varies from 0 to π and ϕ varies from 0 to 2π. Why θ is allowed to vary up to π only and not up to 2π is explained in figure 1.40.

In figure 1.40a to 1.40d θ varies from 0 to π and forms a semicircle. After varying θ from 0 to π and forming a semicircle, ϕ is varied in figure 1.40 e to 1.40 g.

In figure 1.40e we vary ϕ from 0 to 90^0. In figure 1.40f ϕ is varied to 270^0 and in figure 1.40g ϕ completes 2π degrees which finally forms sphere. Thus variation of θ from 0 to π forms a semicircle and when the semicircle is rotated through 2π degrees [ie. we vary ϕ] we finally form a sphere.



z
$\hat{\phi}$
$\hat{\mathbf{r}}$
$\hat{\boldsymbol{\theta}}$
P
θ
r
O
y
ϕ
Q
x
(a)
dr
dl_1
r
(b)
r dθ
r
dθ
dl_2
(c)
r sinθ dϕ
dl_3
r
θ
dϕ
r sin ϕ
(d)
Volume
Element
(e)

Fig.1.39

To form a hemisphere we vary θ from 0 to $\pi/2$ as shown in figure 1.40h –j. Then we rotate ϕ from 0 to 2π in figure 1.40k-m which finally forms a hemisphere.

An elemental length $\mathrm{d}\boldsymbol{l}$ can be expressed in spherical polar coordinates as

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}l_1\,\hat{\mathbf{r}} + \mathrm{d}l_2\,\hat{\boldsymbol{\theta}} + \mathrm{d}l_3\,\hat{\boldsymbol{\phi}} \tag{1.109}$$

Using the result from figure 1.39 b, c, d

$$\mathrm{d}\boldsymbol{l} = \mathrm{dr}\,\hat{\mathbf{r}} + \mathrm{r}\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + \mathrm{r}\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} \tag{1.110}$$

The volume element $\mathrm{d}\tau$ shown in figure 1.39e, 1.41 can be expressed as

$$\mathrm{d}\tau = (\mathrm{d}l_1)\times(\mathrm{d}l_2)\times(\mathrm{d}l_3) \tag{1.111}$$

$$\mathrm{d}\tau = (\mathrm{dr})(\mathrm{r}\,\mathrm{d}\theta)(\mathrm{r}\sin\theta\ \mathrm{d}\phi)$$

$$\mathrm{d}\tau = \mathrm{r}^2\sin\theta\,\mathrm{dr}\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{1.112}$$

Let us calculate the area elements in figure 1.41

$$\left.\begin{aligned} \mathrm{ds}_1 &= \mathrm{d}l_1\times\mathrm{d}l_3 = \mathrm{r}\sin\theta\,\mathrm{dr}\,\mathrm{d}\phi \\ \mathrm{ds}_2 &= \mathrm{d}l_2\times\mathrm{d}l_1 = \mathrm{r}\,\mathrm{dr}\,\mathrm{d}\theta \\ \mathrm{ds}_3 &= \mathrm{d}l_2\times\mathrm{d}l_3 = \mathrm{r}^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \end{aligned}\right\} \tag{1.113}$$

The transformation equation from Cartesian coordinates to spherical coordinates are

$$\left.\begin{aligned} x &= \mathrm{r}\sin\theta\cos\phi \\ y &= \mathrm{r}\sin\theta\sin\phi \\ z &= \mathrm{r}\cos\theta \end{aligned}\right\} \tag{1.114}$$

and

$$\left.\begin{aligned} \mathrm{r} &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \\ \phi &= \tan^{-1}\frac{y}{x} \end{aligned}\right\} \tag{1.115}$$

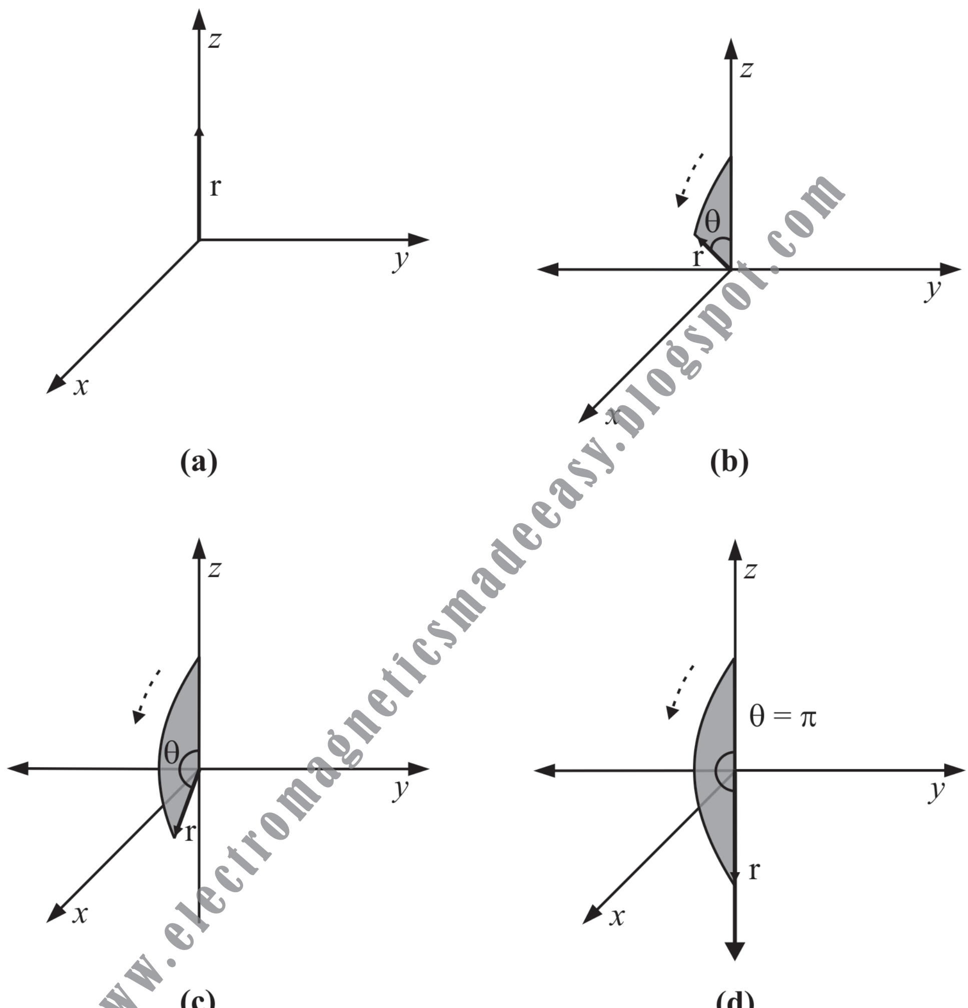
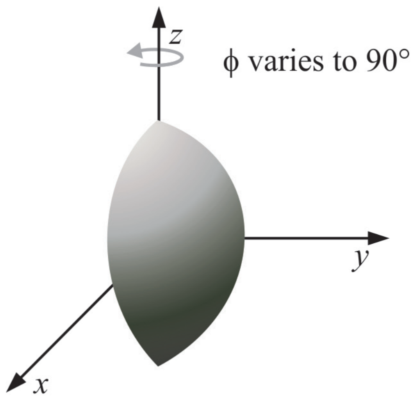
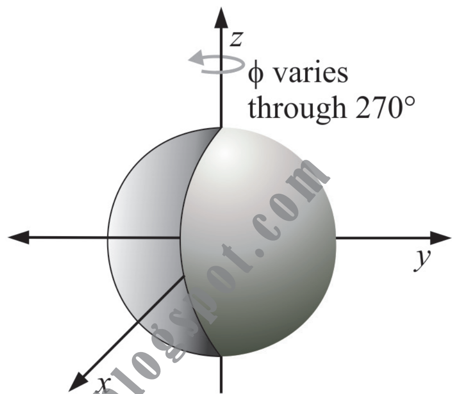


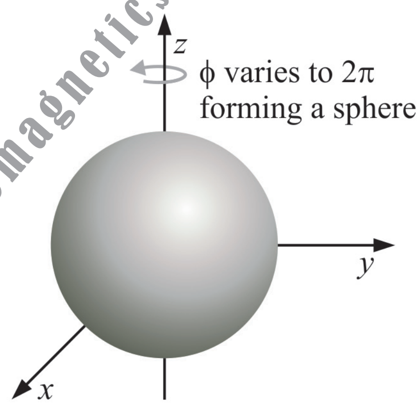
Fig.1.40 (a, b, c, d)



(e)



(f)



(g)

Fig.1.40 (e, f, g)

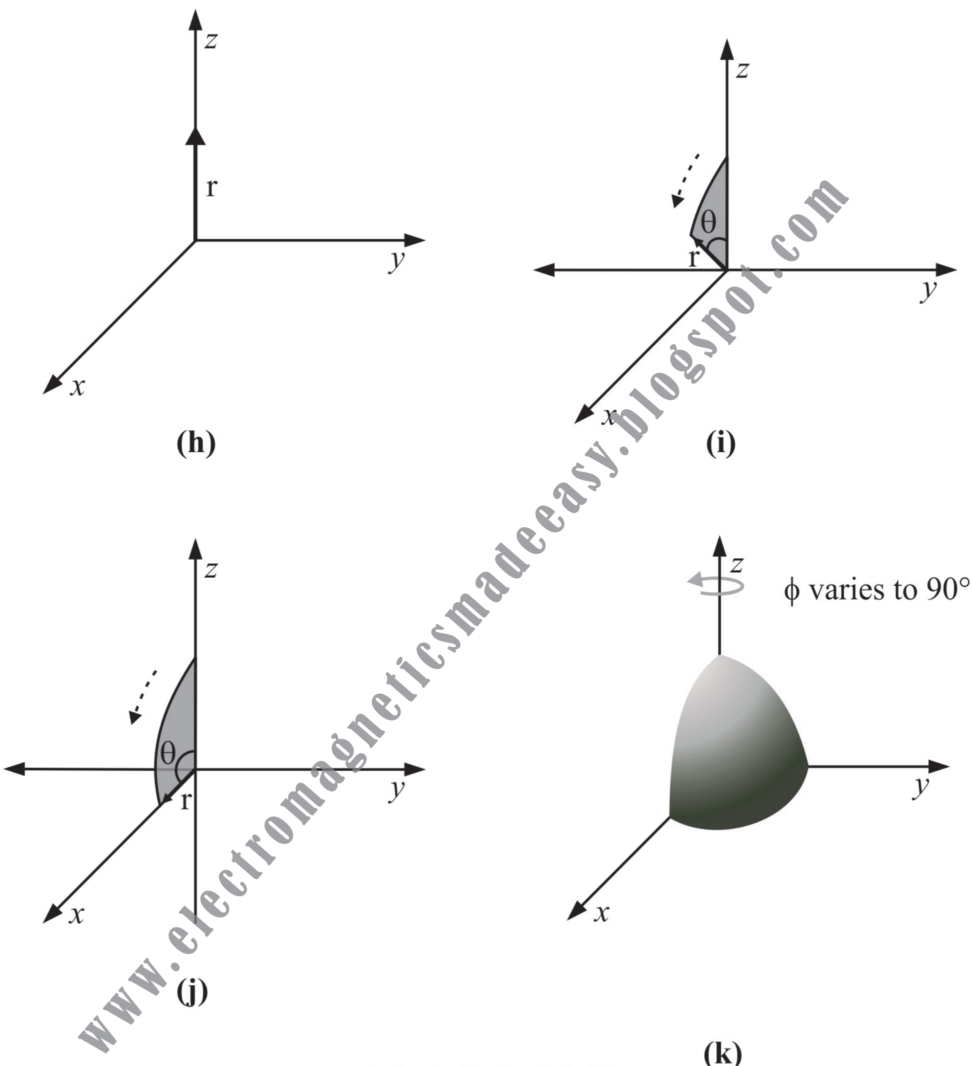
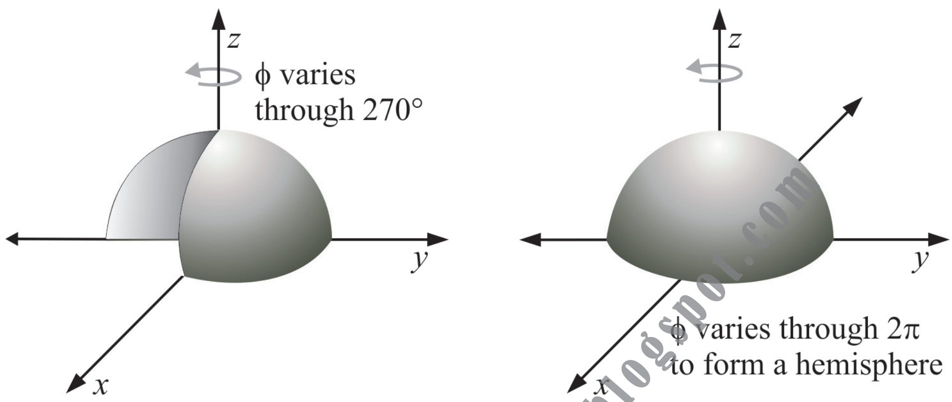


Fig.1.40 (h, i, j, k)



(l) **(m)**

Fig.1.40 (l, m)

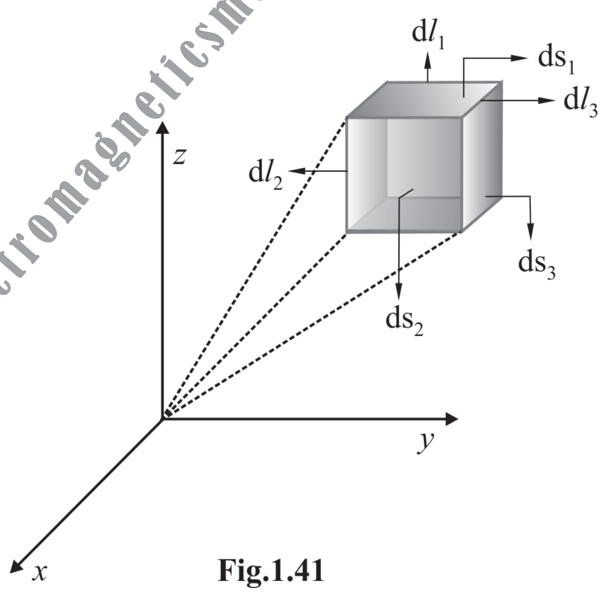


Fig.1.41

Vector Analysis

If u is any scalar function and if **A** is any vector function expressed in spherical coordinates as

$$\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}}$$

then gradient, divergence and curl can be expressed in spherical coordinates as

$$\nabla u = \frac{\partial u}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1.116}$$

$$\nabla . \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 A_r\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin A_\theta\right] + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{1.117}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}}$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{1.118}$$

The ∇^2 operator called Laplacian is given by

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial u}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi} \tag{1.119}$$

1.19b Cylindrical Coordinates

In figure 1.42 cartesian coordinate system along with cylindrical coordinate system (r_C, ϕ, z) is shown. Here r_C is the distance from point P to the *z* axis, ϕ is the angle between *x* axis and line O Q and *z* is the usual cartesian coordinate.

Here r_C varies from 0 to ∞, ϕ varies from 0 to 2π and *z* varies from $-\infty$ to ∞.

An element $\mathrm{d}\boldsymbol{l}$ in cylindrical coordinates can be expressed as

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}l_1\,\hat{\mathbf{r}}_C + \mathrm{d}l_2\,\hat{\boldsymbol{\phi}} + \mathrm{d}l_3\,\hat{\mathbf{z}} \tag{1.120}$$

Using the results from figure 1.43 we can write

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}\,\mathrm{r}_C\,\hat{\mathbf{r}}_C + \mathrm{r}_C\mathrm{d}\phi\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\mathbf{z}} \tag{1.121}$$

The volume element shown in figure 1.44 can be written as

$$\mathrm{d}\tau = \mathrm{d}l_1\,\mathrm{d}l_2\,\mathrm{d}l_3 \tag{1.122}$$

$$\mathrm{d}\tau = \mathrm{r}_C\mathrm{d}\,\mathrm{r}_C\mathrm{d}\phi\mathrm{d}z \tag{1.123}$$

Now let us calculate the area element in figure 1.44.

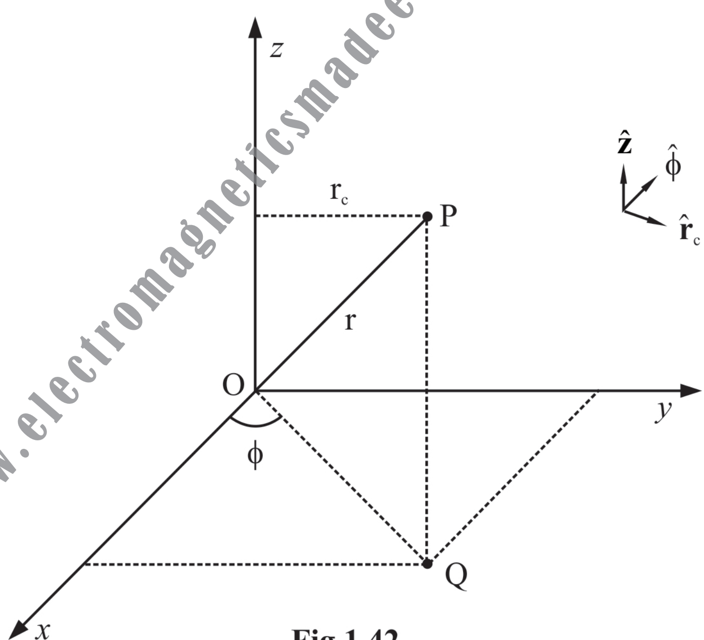


Fig.1.42

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Vector Analysis

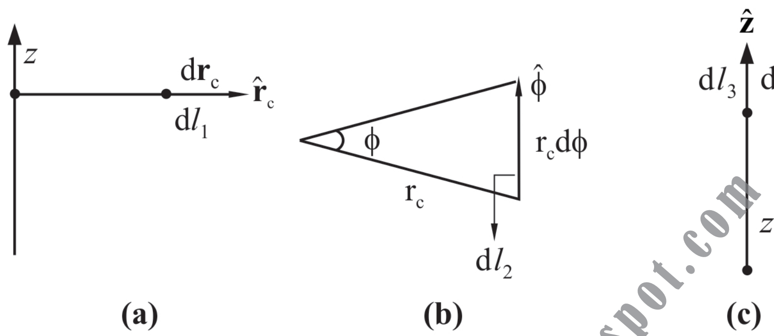

Fig.1.43

$$\left.\begin{aligned} ds_1 &= dl_1\, dl_3 = dr_C\, dz \\ ds_2 &= dl_1\, dl_2 = r_C dr_C\, d\phi \\ ds_3 &= dl_2\, dl_3 = r_C dr_C\, dz \end{aligned}\right\} \tag{1.124}$$

The transformation equation from cartesian to cylindrical coordinate system is given by

$$x = r_C \cos\theta, y = r_C \sin\phi, z = z \tag{1.125}$$

also

$$\left.\begin{aligned} r_C &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1}\frac{y}{x} \\ z &= z \end{aligned}\right\} \tag{1.126}$$

If u is any scalar function and if any vector **A** is expressed in cylindrical coordinates as

$$\mathbf{A} = A_{r_C}\,\hat{\mathbf{r}}_C + A_\phi\,\hat{\boldsymbol{\phi}} + A_z\,\hat{\mathbf{z}}$$

Then the gradient, divergence and curl can be expressed in cylindrical coordinates as

$$\nabla u = \frac{\partial u}{\partial r}\hat{\mathbf{r}}_C + \frac{1}{r_C}\frac{\partial u}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial u}{\partial z}\hat{\mathbf{z}} \tag{1.127}$$

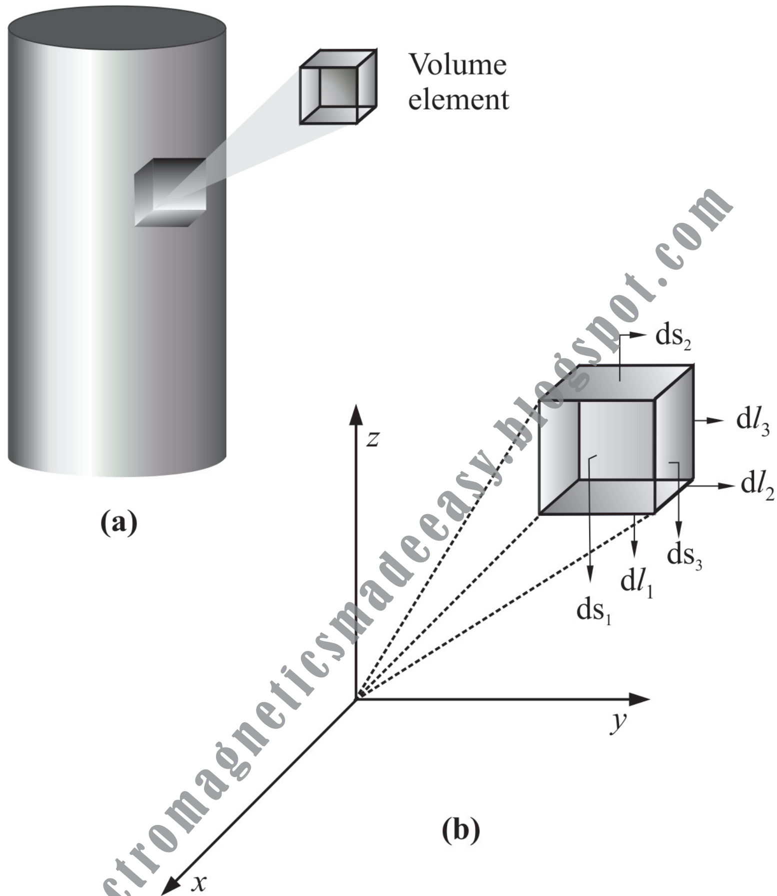

Fig.1.44

$$\nabla.\mathbf{A} = \frac{1}{r_C}\frac{\partial}{\partial r_C}(r_C A_{r_C}) + \frac{1}{r_C}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{1.128}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r_C}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\mathbf{r}}_C + \left(\frac{\partial A_{r_C}}{\partial z} - \frac{\partial A_z}{\partial r_C}\right)\hat{\phi}$$

$$+\frac{1}{r_C}\left(\frac{\partial}{\partial r}(r_C A_\phi) - \frac{\partial A_{r_C}}{\partial \phi}\right)\hat{\mathbf{z}} \tag{1.129}$$

The Laplacian ∇^2 is given by

$$\nabla^2 u = \frac{1}{r_C}\frac{\partial}{\partial r_C}\left(r_C\frac{\partial u}{\partial r_C}\right) + \frac{1}{r_C^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} \tag{1.130}$$

Example 1.12

Convert the cartesian coordinates (3, 4, 5) into spherical coordinates.

Solution:

(a) In spherical coordinates

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$r = \sqrt{3^2 + 4^2 + 5^2} = 7.07$$

$$\theta = \tan^{-1}\frac{\sqrt{3^2 + 4^2}}{5} = 45°$$

$$\phi = \tan^{-1}\frac{4}{3} = 53.13°$$

1.20 Important Vector Identities

Let u be a scalar function **A**, **B** be vector functions. The following are few important vector identities.

1. $\nabla(\mathbf{A}.\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}.\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{B}.\nabla)\mathbf{A}$
2. $\text{curl}(\mathbf{A}+\mathbf{B}) = \text{curl}\,\mathbf{A} + \text{curl}\,\mathbf{B}$
3. $\text{div}\,\text{curl}\,\mathbf{A} = 0$
4. $\text{curl}\,\text{grad}\,u = 0$

5. $\text{div}(\mathbf{A}+\mathbf{B}) = \text{div}\,\mathbf{A} + \text{div}\,\mathbf{B}$
6. $\text{curl}(\text{u}\,\mathbf{A}) = \text{u}\,\text{curl}\,\mathbf{A} + \text{grad}\,\text{u}\times\mathbf{A}$
7. $\text{curl}\,\text{curl}\,\mathbf{A} = \text{grad}\,\text{div}\,\mathbf{A} - \nabla^2\mathbf{A}$
8. $\text{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}.\text{curl}\,\mathbf{A} - \mathbf{A}.\text{curl}\,\mathbf{B}$
9. $\text{curl}(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}.\nabla)\mathbf{A} - (\mathbf{A}.\nabla)\mathbf{B} + \mathbf{A}(\nabla.\mathbf{B}) - \mathbf{B}(\nabla.\mathbf{A})$
10. $\text{div}(\text{u}\,\mathbf{A}) = \text{u}\,\text{div}\,\mathbf{A} + \mathbf{A}.\text{grad}\,\text{u}$
11. $\nabla.\nabla = \left(\hat{\mathbf{i}}\frac{\partial}{\partial \text{x}} + \hat{\mathbf{j}}\frac{\partial}{\partial \text{y}} + \hat{\mathbf{k}}\frac{\partial}{\partial \text{z}}\right).\left(\hat{\mathbf{i}}\frac{\partial}{\partial \text{x}} + \hat{\mathbf{j}}\frac{\partial}{\partial \text{y}} + \hat{\mathbf{k}}\frac{\partial}{\partial \text{z}}\right)$

$$\Rightarrow \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ is called Laplacian operator.}$$

The above form of Laplacian operator is in cartesian coordinates. We have seen the form of Laplacian in spherical coordinates in section 1.19a and in cylindrical coordinats in section 1.19b.

Example 1.13

Calculate ∇r if $\text{r} = (x^2 + y^2 + z^2)^{1/2}$ where $\hat{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Solution:

$$\nabla \text{r} = \hat{\mathbf{i}}\frac{\partial \text{r}}{\partial x} + \hat{\mathbf{j}}\frac{\partial \text{r}}{\partial y} + \hat{\mathbf{k}}\frac{\partial \text{r}}{\partial z}$$

$$\nabla \text{r} = \hat{\mathbf{i}}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} + \hat{\mathbf{j}}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{1/2}$$

$$+\hat{\mathbf{k}}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{1/2}$$

$$= \hat{\mathbf{i}}\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}.2x$$

$$+\hat{\mathbf{j}}\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)$$

$$+\hat{\mathbf{k}}\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)$$

$$=(x^2+y^2+z^2)^{-1/2}(\hat{\mathbf{i}}\,x+\hat{\mathbf{j}}\,y+\hat{\mathbf{k}}\,z)$$

$$=\frac{\mathbf{r}}{\mathrm{r}}=\frac{\mathrm{r}\,\hat{\mathbf{r}}}{\mathrm{r}}=\hat{\mathbf{r}}$$

Example 1.14

Show that $\nabla\left(\frac{1}{\mathrm{r}}\right)=\frac{-\mathbf{r}}{\mathrm{r}^3}=\frac{-\hat{\mathrm{r}}}{\mathrm{r}^2}$

where $\mathbf{r}$ is the position vector of a point $\mathbf{r}=\hat{\mathbf{i}}\,x+\hat{\mathbf{j}}\,y+\hat{\mathbf{k}}\,z$

Solution:

$$\nabla\frac{1}{\mathrm{r}}=\hat{\mathbf{i}}\frac{\partial}{\partial x}\left(\frac{1}{\mathrm{r}}\right)+\hat{\mathbf{j}}\frac{\partial}{\partial y}\left(\frac{1}{\mathrm{r}}\right)+\hat{\mathbf{k}}\frac{\partial}{\partial z}\left(\frac{1}{\mathrm{r}}\right)$$

with $\mathrm{r}^2 = x^2 + y^2 + z^2$

$$\frac{\partial}{\partial x}\left(\frac{1}{\mathrm{r}}\right)=\frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)^{1/2}}\right)$$

$$=\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{-1/2}\right]$$

$$=\frac{-1}{2}\,\frac{2x}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{-x}{\left((x^2+y^2+z^2)^{\frac{3}{2}}\right)} = \frac{-x}{\mathrm{r}^3}$$

similarly $\dfrac{\partial}{\partial y}\left(\dfrac{1}{\mathrm{r}}\right) = \dfrac{-y}{\mathrm{r}^3}$

$$\frac{\partial}{\partial y}\left(\frac{1}{\mathrm{r}}\right) = \frac{-z}{\mathrm{r}^3}$$

Thus $\boldsymbol{\nabla}\left(\dfrac{1}{\mathrm{r}}\right) = \hat{\mathbf{i}}\left(\dfrac{-x}{\mathrm{r}^3}\right) + \hat{\mathbf{j}}\left(\dfrac{-y}{\mathrm{r}^3}\right) + \hat{\mathbf{k}}\left(\dfrac{-z}{\mathrm{r}^3}\right)$

$$\boldsymbol{\nabla}\left(\frac{1}{\mathrm{r}}\right) = \frac{-(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{\mathrm{r}^3} = \frac{-\mathbf{r}}{\mathrm{r}^3}$$

$$\Rightarrow \boldsymbol{\nabla}\left(\frac{1}{\mathrm{r}}\right) = \frac{-\mathbf{r}}{\mathrm{r}^3} = \frac{-\mathrm{r}\,\hat{\mathbf{r}}}{\mathrm{r}^3} = \frac{-\hat{\mathbf{r}}}{\mathrm{r}^2}$$

1.21 Two and Three Dimensions

Throughout electromagnetics we will be discussing about two and three dimensional problems. If the reader has difficulty in visualizing two and three dimensions go to a corner of the room as shown in figure 1.45. In that figure, O is the corner of the room. Fix O, the corner of the room as the origin. Two walls along with the floor is shown in the same figure. Take the line joining the walls and floor as x, y, z axis. Floor is the $x - y$ plane. Wall 1 is the $x - z$ plane, wall 2 is the $z - y$ plane. We have shown an ant moving on $x - z$ plane, that is a wall 1. If we want to describe the motion of the ant then we need x, z axis alone. No need for y axis as long as the ant moves in $x - z$ plane. Thus the problem is two dimensional. On the

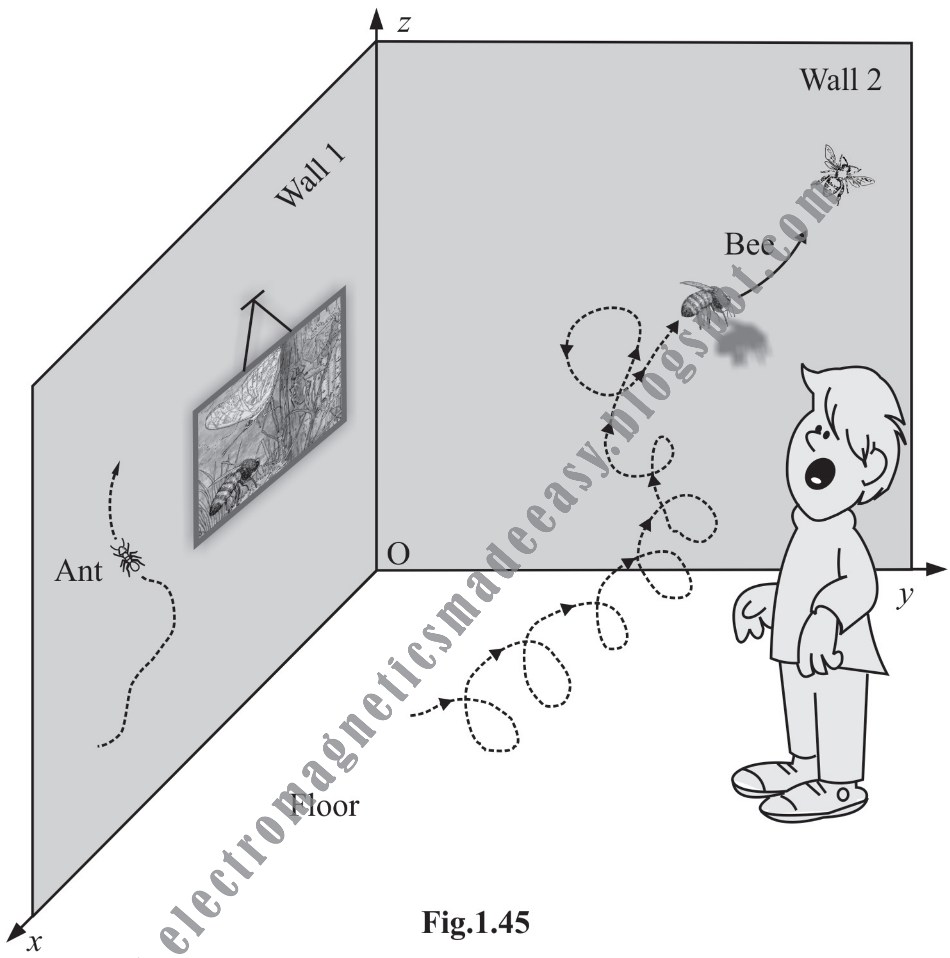

Fig.1.45

other hand if a bee is flying inside the room as shown in the same figure then we need all the three axis x, y, z to describe the motion of bee then the problem is three dimensional.

Example 1.15

Prove that

$$-\int_S \nabla \mathrm{u} \times d\mathbf{s} = \oint_C \mathrm{u}\, d\boldsymbol{l}$$

where u is a scalar.

Solution:

Consider any vector **G**, from Stoke's theorem

$$\int_S (\nabla\times\mathbf{G}).d\mathbf{s} = \oint_C \mathbf{G}.d\boldsymbol{l}$$

Let us assume that **G** can be written as **G = u N** where u is a scalar function and **N** is a constant vector. Then

$$\int (\nabla\times u\,\mathbf{N}).d\mathbf{s} = \oint_C u\,\mathbf{N}.d\boldsymbol{l} \qquad (1.131)$$

From section 1.20 using vector identity 6

$$\nabla\times u\,\mathbf{N} = u\nabla\times\mathbf{N} + \nabla u\times\mathbf{N}$$

But $\nabla\times\mathbf{N} = 0$ because **N** is a constant vector. Hence

$$\nabla\times u\,\mathbf{N} = \nabla u\times\mathbf{N} \qquad (1.132)$$

Substituting equation 1.132 in equation 1.131 we get

$$\int (u\nabla\times\mathbf{N}).d\mathbf{s} = \oint_C u\,\mathbf{N}.d\boldsymbol{l}$$

$$\Rightarrow -\int (\mathbf{N}\times\nabla u).d\mathbf{s} = \oint_C u\,\mathbf{N}.d\boldsymbol{l}$$

$$\Rightarrow -\int (\nabla u\times d\mathbf{s}).\mathbf{N} = \oint_C u\,\mathbf{N}.d\boldsymbol{l}$$

As **N** is a constant vector, pulling **N** out of integral and cancelling both sides.

$$\Rightarrow -\int_S \nabla u\times d\mathbf{s} = \oint_C u\; d\boldsymbol{l} \qquad (1.133)$$

Hence proved.

Example 1.16

Prove that for any two vector **Q**, **R** where **Q** is a constant vector

$$\nabla\times(\mathbf{Q}\times\mathbf{R})=\mathbf{Q}(\nabla.\mathbf{R})-(\mathbf{Q}.\nabla)\mathbf{R}$$

Solution:

From section 1.20 using vector identity 9 we get

$$\nabla\times(\mathbf{Q}\times\mathbf{R})=(\mathbf{R}.\nabla)\mathbf{Q}-(\mathbf{Q}.\nabla)\mathbf{R}$$
$$+\mathbf{Q}(\nabla.\mathbf{R})-\mathbf{R}(\nabla.\mathbf{Q})$$

As **Q** is a constant vector

$$\nabla.\mathbf{Q}=0,(\mathbf{R}.\nabla)\mathbf{Q}=0$$

Hence

$$\nabla\times(\mathbf{Q}\times\mathbf{R})=\mathbf{Q}(\nabla.\mathbf{R})-(\mathbf{Q}.\nabla)\mathbf{R} \qquad (1.134)$$

Example 1.17

For a constant vector **Q** and scalar function u prove that

$$\mathbf{S}\times\mathbf{Q}=\oint_{C}\mathbf{Q}.\mathbf{r}\,\mathrm{d}l$$

where S is the surface bounded by loop C and $\mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$.

Solution:

Let u = **Q** . **r** where **Q** is a constant vector. From section 1.20 using vector identity 1

$$\nabla u=\nabla(\mathbf{Q}.\mathbf{r})=\mathbf{Q}\times(\nabla\times\mathbf{r})+(\mathbf{Q}.\nabla)\mathbf{r}$$
$$+\mathbf{r}\times(\nabla\times\mathbf{Q})+(\mathbf{r}.\nabla)\mathbf{Q}$$

Because **Q** is a constant vector

$$\nabla\times\mathbf{Q}=0 \text{ and } (\mathbf{r}.\nabla)\mathbf{Q}=0$$

Hence

$$\nabla u=\nabla(\mathbf{Q}.\mathbf{r})=\mathbf{Q}\times(\nabla\times\mathbf{r})+(\mathbf{Q}.\nabla)\mathbf{r}$$

But $\nabla \times \mathbf{r} = 0$. Hence

$$\nabla \mathrm{u} = \nabla(\mathbf{Q}.\mathbf{r}) = (\mathbf{Q}.\nabla)\mathbf{r}$$

$$= \left(\mathrm{Q}_x \frac{\partial}{\partial x} + \mathrm{Q}_y \frac{\partial}{\partial y} + \mathrm{Q}_z \frac{\partial}{\partial z}\right).\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)$$

$$= \mathrm{Q}_x \hat{\mathbf{i}} + \mathrm{Q}_y \hat{\mathbf{j}} + \mathrm{Q}_z \hat{\mathbf{k}} = \mathbf{Q}$$

Subsituting $\nabla \mathrm{u} = \mathbf{Q}$ and $\mathrm{u} = (\mathbf{Q}.\mathbf{r})$ in equation 1.133

$$-\int_S \mathbf{Q} \times \mathrm{d}\mathbf{s} = \oint_C (\mathbf{Q}.\mathbf{r})\,\mathrm{d}\boldsymbol{l}$$

Because **Q** is a constant vector

$$-\mathbf{Q} \times \int_S \mathrm{d}\mathbf{s} = \oint_C (\mathbf{Q}.\mathbf{r})\,\mathrm{d}\boldsymbol{l} \qquad (1.135)$$

$$\Rightarrow -\mathbf{Q} \times \mathbf{S} = \oint_C (\mathbf{Q}.\mathbf{r})\,\mathrm{d}\boldsymbol{l}$$

$$\Rightarrow \mathbf{S} \times \mathbf{Q} = \oint_C (\mathbf{Q}.\mathbf{r})\,\mathrm{d}\boldsymbol{l} \qquad (1.136)$$

Vector Analysis

EXERCISES

Problem 1.1

Check whether the two vectors $\mathbf{P} = 12\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\mathbf{Q} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + z\hat{\mathbf{k}}$ are perpendicular to each other.

Problem 1.2

If $\mathbf{P} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ find **Q** if **Q** is parallel to **P**.

Problem 1.3

Check whether the vectors $\mathbf{P} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 18\hat{\mathbf{k}}$,

$\mathbf{Q} = 9\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\mathbf{R} = 10\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 15\hat{\mathbf{k}}$ are coplanar.

Problem 1.4

If $\mathbf{P} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ find the unit vector in the direction of **P**.

Problem 1.5

Check whether the three vectors $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$,

and $\mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}, \mathbf{C} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ form a right angled triangle.

Problem 1.6

A vector **P** is directed from (3, 4, 5) to (1, 3, 2). Determine $|\mathbf{P}|$ and unit vector in the direction of **P**.

Problem 1.7

Find the angle between the two vectors $\mathbf{P} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{Q} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ using dot product and cross product.

Problem 1.8

Find the value of a such that the three vectors $10\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$,

$2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + a\hat{\mathbf{k}}$ are coplanar.

Problem 1.9

Find the projection of vector $\mathbf{P} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ on vector $\mathbf{Q} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

Problem 1.10

If $\mathbf{P} + \mathbf{Q} + \mathbf{R} = 0$ then show that $\mathbf{P} \times \mathbf{Q} = \mathbf{Q} \times \mathbf{R} = \mathbf{R} \times \mathbf{P}$.

Problem 1.11

Calculate $(\mathbf{P} \times \mathbf{Q}) . (\mathbf{R} \times \mathbf{S})$

Problem 1.12

Show that $\nabla \log \mathrm{r} = \frac{\mathbf{r}}{\mathrm{r}^2}$ where $\mathbf{r}$ is the position vector.

Problem 1.13

If $\mathbf{r}$ is the position vector show that

(i) $\nabla \mathrm{r}^{\mathrm{n}} = \mathrm{r}\,\mathrm{n}^{\mathrm{n-2}}\mathbf{r}$

(ii) $\mathrm{div}\,\mathbf{r} = 3$

(iii) $\mathrm{div}\,\mathrm{r}^{\mathrm{n}}\mathbf{r} = (\mathrm{n}+3)\mathrm{r}^{\mathrm{n}}$

(iv) $\mathrm{curl}(\mathrm{r}^{\mathrm{n}}\mathbf{r}) = 0$

Problem 1.14

Calculate $\nabla \mathrm{u}$ at $(1, -5, 3)$ if $\mathrm{u} = x^2 z + y^3 z^2$.

Problem 1.15

Show that $\nabla.(\mathbf{P}+\mathbf{Q}) = \nabla.\mathbf{P} + \nabla.\mathbf{Q}$

Problem 1.16

Calculate a if $\mathbf{A} = (3x+y)\hat{\mathbf{i}} + (z-x)\hat{\mathbf{j}} + (\mathrm{a}y+z)\hat{\mathbf{k}}$ is solenoidal.

Problem 1.17

If $\mathbf{P} = 3x^2\,\hat{\mathbf{i}} - y^2 z\,\hat{\mathbf{j}} + 2x^3 z\,\hat{\mathbf{k}}$ find

(a) $\nabla \times \mathbf{P}$

(b) $\nabla \times (\nabla \times \mathbf{P})$

Problem 1.18

If $\mathbf{P} = (2x^2 + y)\,\hat{\mathbf{i}} - 10\,yz\,\hat{\mathbf{j}} + 12\,xz^2\,\hat{\mathbf{k}}$

Calculate $\int \mathbf{P}.d\mathbf{r}$ along the straight line from (0,0,0) to (1, 1, 0) and then to (1,1,1).

Problem 1.19

Calculate the gradients of

(a) $u(r,\theta,\phi) = 3r\sin\theta - 4\phi + 1$

(b) $u(r,\theta,z) = 3\cos\phi - rz$

Problem 1.20

Calculate div **P** if

(a) $\mathbf{P} = 3\hat{\mathbf{r}} + r\sin\theta\hat{\boldsymbol{\theta}} + r\hat{\boldsymbol{\phi}}$

(b) $\mathbf{P} = r\hat{\mathbf{r}} + z\cos\phi\hat{\boldsymbol{\phi}} + 3\hat{\mathbf{z}}$

Problem 1.21

Calculate the net flux of the vector field $\mathbf{P}(x,y,z) = 3xy^2\,\hat{\mathbf{i}} + z^2\,\hat{\mathbf{j}} + y^3\hat{\mathbf{k}}$ emerging from a cube of dimensions $0 \le x,y,z \le 1$

Problem 1.22

Solve Problem 1.21 using Gauss divergence theorem.

Problem 1.23

Verity divergence theorem for the vector $\mathbf{P} = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$ for the cube $0 \le x,y,z \le 1$ shown in figure 1.25.

Problem 1.24

Verify Stoke's theorem for the vector $\mathbf{P} = (x+y)\hat{\mathbf{i}} + (y+z)\hat{\mathbf{j}} + (x+z)\hat{\mathbf{k}}$ for a plane rectangular area with vertices (0,0), (2,0), (2,1), (0,1) as shown in figure 1.46

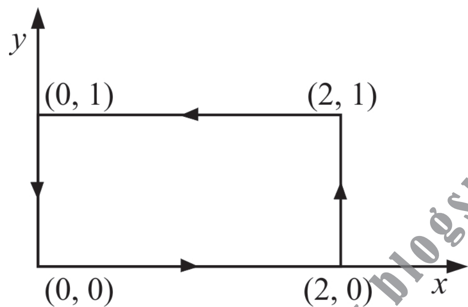

Fig.1.46

Problem 1.25

Find the value of a, b and c if

$\mathbf{P} = (4x + 5y + \mathrm{a}\,z)\hat{\mathbf{i}} + (3x + \mathrm{b}\,y + z)\hat{\mathbf{j}} + (\mathrm{c}\,x + 2y - 3z)\hat{\mathbf{k}}$ is irrotational.

Problem 1.26

Calculate the surface area and volume of a sphere by integrating surface and volume elements in spherical polar coordinates.

Problem 1.27

Calculate the surface area and volume of a cylinder by integrating surface and volume elements in a cylindrical coordinates.

Problem 1.28

Prove that for any vector **G** $\int_{\tau} \nabla \times \mathbf{G}\, d\tau = -\oint_{S} \mathbf{G} \times d\mathbf{s}$.